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Excursions in Summation

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Using $\arctan(x) + \frac{\pi}{2}$ to approximate functions on $C[-1, 1]$

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Introduction

This project concerns approximation properties of the set

$$S(\varphi, X) := \left\{ \sum_{j=1}^{N} a_j \varphi(x - x_j) : a_j \in \mathbb{R}, x_j \in X \right\},$$

where X is a scattered sequence and $\varphi(x) = \arctan(x-y) + \frac{\pi}{2}$. Similar approximation sets are commonly used in interpolation problems and are especially helpful due to their Fourier representation. For our work, we will work to prove the following theorem. Suppose $f \in C[a, b]$. For any $\epsilon > 0$, there exists $s \in S$, such that

$$||f - s||_{L\infty} < \epsilon$$

We begin with the Taylor Series for

$$\phi(x-y) =: \arctan(x-y) + \frac{\pi}{2}$$

giving us

$$\phi(x-y) =: \sum_{j=0}^{\infty} \frac{B_j(x)}{y^j}$$

where $B_j(x)$ is a polynomial and linear algebra allows us to actually recover these polynomials.

Vandermonde Matrix

A Vandermonde matrix is a matrix where the power of the values increase as you move down the column or as you move across the row. In the $N \times N$ for my problem it looks like

$$\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{y_1} & \frac{1}{y_2} & \frac{1}{y_3} & \cdots & \frac{1}{y_N} \\ \frac{1}{y_1^2} & \frac{1}{y_2^2} & \frac{1}{y_3^2} & \cdots & \frac{1}{y_N^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{y_1^N} & \frac{1}{y_2^N} & \frac{1}{y_3^N} & \cdots & \frac{1}{y_N^N} \end{bmatrix}$$

The determinant of a square Vandermonde matrix can be expressed as

$$\det(V) = \prod_{1 \le i < j \le n} \left(\frac{1}{y_j} - \frac{1}{y_i} \right).$$

Cramer's Rule

Cramer's Rule is a way of solving for coefficients in a square system using the determinant of the original system and the determinant of the system with the n^{th} column swapped with the solution set. So in this 3×3 example,

$$\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{y_1} & \frac{1}{y_2} & \frac{1}{y_3} \\ \frac{1}{y_1^2} & \frac{1}{y_2^2} & \frac{1}{y_3^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

You would find each coefficient using the respective determinants

$$a_1 = \frac{\det \begin{bmatrix} 0 & 1 & 1 \\ 0 & \frac{1}{y_2} & \frac{1}{y_3} \\ 1 & \frac{1}{y_2^2} & \frac{1}{y_3^2} \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{y_1} & \frac{1}{y_2} & \frac{1}{y_3} \\ \frac{1}{y_1^2} & \frac{1}{y_2^2} & \frac{1}{y_3^2} \end{bmatrix}} \qquad a_2 = \frac{\det \begin{bmatrix} 1 & 0 & 1 \\ \frac{1}{y_1} & 0 & \frac{1}{y_3} \\ \frac{1}{y_1^2} & 1 & \frac{1}{y_3^2} \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{y_1} & \frac{1}{y_2} & \frac{1}{y_3} \\ \frac{1}{y_1^2} & \frac{1}{y_2^2} & \frac{1}{y_3^2} \end{bmatrix}} \qquad a_3 = \frac{\det \begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{y_1} & \frac{1}{y_2} & 0 \\ \frac{1}{y_1^2} & \frac{1}{y_2^2} & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{y_1} & \frac{1}{y_2} & \frac{1}{y_3} \\ \frac{1}{y_1^2} & \frac{1}{y_2^2} & \frac{1}{y_3^2} \end{bmatrix}}$$

$\varphi(x) = \arctan(x) + \frac{\pi}{2}$

The following result shows that our polynomials span and are linearly independent in the set of natural numbers. Let

$$\arctan(x-y) + \frac{\pi}{2} =: \sum_{k=1}^{\infty} \frac{B_k(x)}{y^k}$$

Then

$$\begin{aligned} &\frac{d}{dy}\left(\sum_{k=1}^{\infty} \frac{B_k(x)}{y^k} \right) \\ &= \frac{-1}{1 + (x-y)^2} \\ &= -\sum_{j=0}^{\infty} \frac{A_j(x)}{y^{j+2}} \end{aligned}$$

So

$$\sum_{k=1}^{\infty} \frac{B_k(x)}{y^k} = -\int \left(\sum_{j=0}^{\infty} \frac{A_j(x)}{y^{j+2}} \right) dy + C$$

$$\sum_{k=1}^{\infty} \frac{B_k(x)}{y^k} = -\sum_{j=0}^{\infty} \frac{A_j(x) y^{-j-1}}{-j-1}$$

And from [1] we know that

$$A_n(x) = (n+1)x^n + \text{ lower order terms}$$

And using those results we can simplify

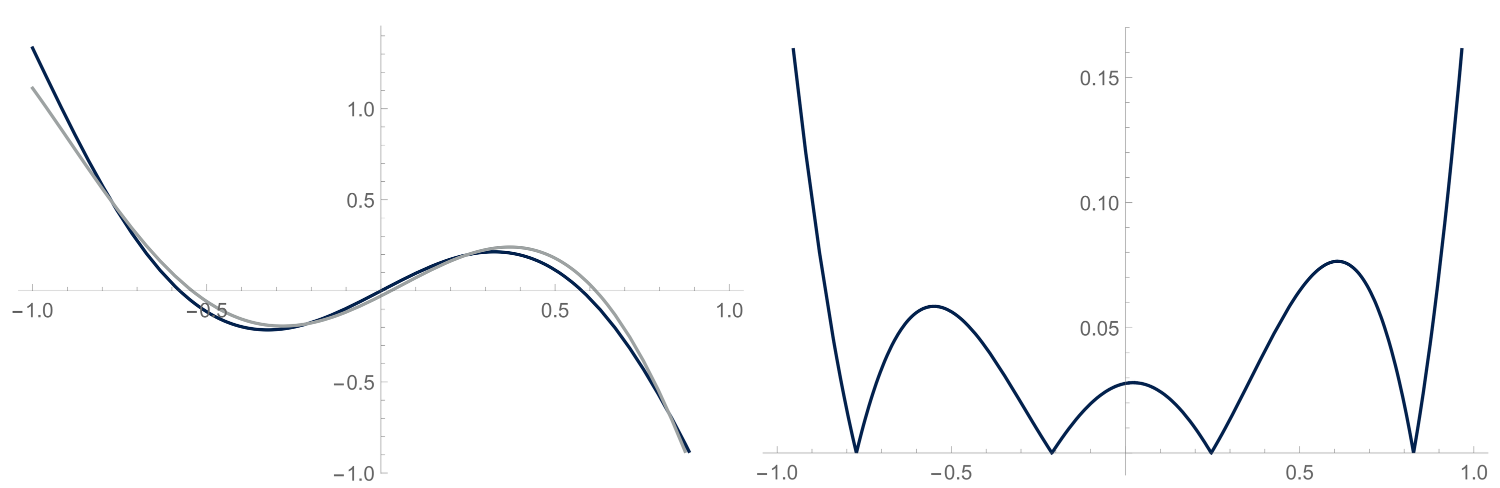
$$\begin{aligned} B_k(x) &= \frac{1}{k} A_{k-1}(x) \\ &= \frac{1}{k} k x^{k-1} + \text{lower order terms} \\ &= x^{k-1} + \text{lower order terms} \end{aligned}$$

Therefore we know our polynomials form a basis for Π. Using the coefficients from the Vandermonde system we have

$$B_{n-1}(x) + O\left(\frac{1}{y_1}\right) = \sum_{j=1}^{N} a_j \phi(x - y_j).$$

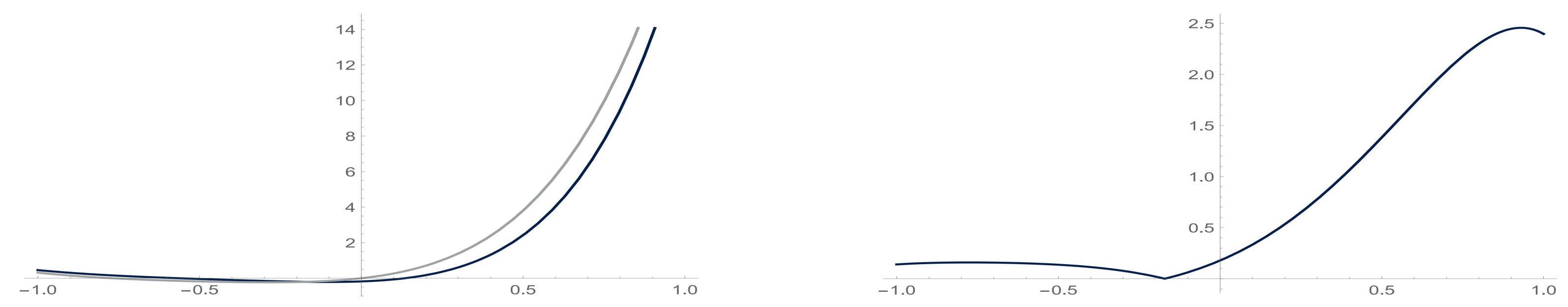
Approximations and Differences in Functions

5th degree approximation and difference between polynomials.

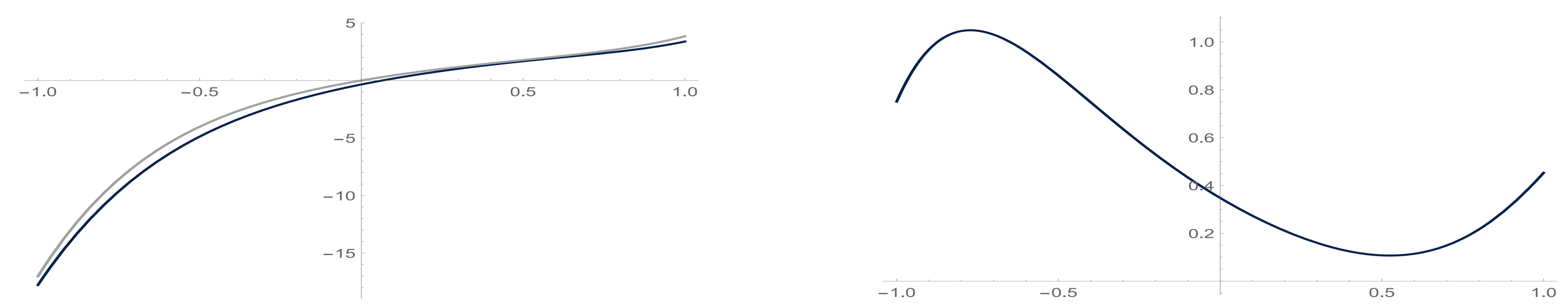


Examples

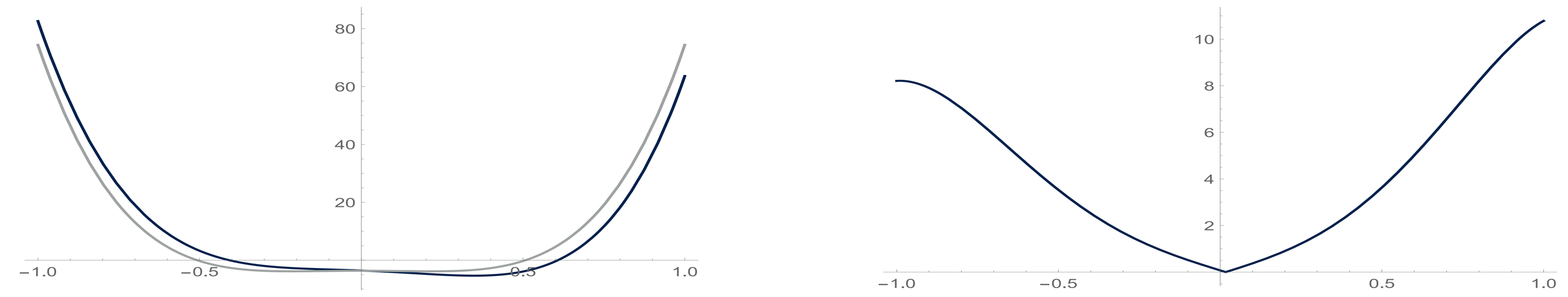
$e^{3x} \sin(2x)$



$x e^{\pi(x+2)^{-1}}$



$4\cos(3\sin(2\cos(2\sin(x))))$



Conclusion

As shown in my work we know that our polynomials $B_j(x)$ form a basis for Π, therefore, $\arctan(x-y) + \frac{\pi}{2}$ can be used for its approximating properties. Furthermore, it is true that this scheme works for arbitrary values of q and r , where r is any real number that is not a natural number and q is a natural number.

$$(1 + x^q)^r$$

Acknowledgements & References

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[2] J.Ledford, On the density of scattered translates of the general multiquadratic in C([a,b]). New York J. Math. **20** (2014), 145–151.

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