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Non-Local Approximation

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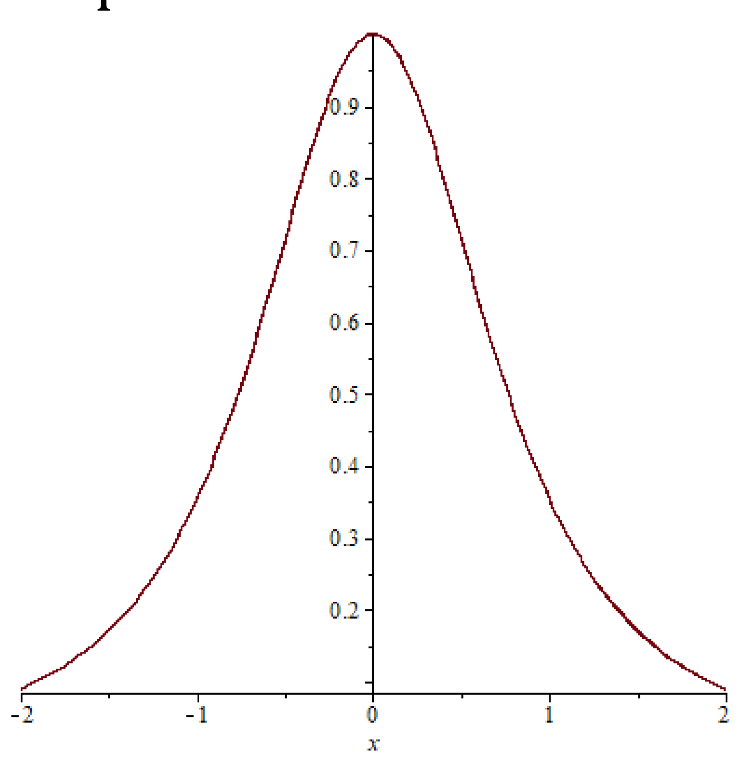
Introduction

This project is a continuation of work previously done by Dr. Jeff Ledford. This research extends the non-local approximation schemes found in [1,2,3]. We seek approximands of the form $C_1\phi(x-x_0)+C_2\phi(x-x_1)+\cdots+C_n\phi(x-x_{n-1})$ which approximate continuous functions uniformly on closed intervals and interpolate the data $\{(x_j,y_j): j=0,1,\dots,n-1\}$. Our function ϕ is taken to be the (general) multiquadric $\phi(x)=(1+x^2)^{-3/2}$ pictured below. This research is still ongoing.



Methods

We first expand ϕ(x-y) in a Taylor series in $\frac{1}{y^n}$, this yields coefficient polynomials $A_n(x)$. If the set of these polynomials span the space of all polynomials, $\Pi[x]$, then the uniform approximation property will follow from the Stone-Weierstrass theorem. A sufficient condition for spanning $\Pi[x]$ involves showing that the leading coefficients of $A_n(x)$ are non-zero.

Proof

We used Maple to generate the first few coefficient polynomials.

$$1,\ 3x$$
$$2,\ 6x^2-\frac{3}{2}$$
$$3,\ 10x^3-\frac{15}{2}x$$
$$4,\ 15x^4-\frac{45}{2}x^2+\frac{15}{8}$$
$$5,\ 21x^5-\frac{105}{2}x^3+\frac{105}{8}x$$
$$6,\ 28x^6-105x^4+\frac{105}{2}x^2-\frac{35}{16}$$
$$7,\ 36x^7-189x^5+\frac{315}{2}x^3-\frac{315}{16}x$$
$$8,\ 45x^8-315x^6+\frac{1575}{4}x^4-\frac{1575}{16}x^2+\frac{315}{128}$$

The leading coefficients seems to be $a_n=\binom{n+2}{2}$. The proof showing this pattern was then sought. We have

$$a_n=(-2)^{-n}\sum_{j=\lceil\frac{n}{2}\rceil}^{n}4^j\binom{-3/2}{j}\binom{j}{2j-n},$$

splitting our problem into two cases, depending on the parity of n, allows us to use a Lemma 1 in [2] to show that $a_n\neq 0$. When n is odd, we can reduce the expression to

$$a_{2M+1}=\frac{(-1)^{M+1}}{2^M M!}\binom{2M+3}{2}\sum_{j=0}^{M}(-1)^j\binom{M}{j}Q(j)$$

where

$$Q(x)=\frac{(2x+1)(2x+3)\cdots(2x+2M+1)}{(M+1)(2M+1)}$$

which is an M^th degree polynomial, whose leading coefficient is 2^M. Now we may use Lemma 1 in [2] which provides,

Lemma 1. For $N\in\mathbb{N}$, $0\le l\le N$, and p a polynomial of degree l. We have,

$$\sum_{j=0}^{N}(-1)^j\binom{N}{j}p(j)=\begin{cases}0 & 0\le l<N\\ (-1)^N a_N\cdot N! & l=N\end{cases}$$

where a_N is the leading coefficient of p.

$$(-1)^{M+1}2^M M!\frac{(-1)^M}{2^M M!}\binom{2M+3}{2}=\binom{2M+3}{2}$$

which is the desired result when $n=2M+1$. The even case is virtually nearly identical.

To clean up this argument, we extended Lemma 1 in [2] to include polynomials of degree $M+1$.

Lemma 2. Suppose $M\in\mathbb{N}$, then we have

$$\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^{M+1}=(-1)^M M!\binom{M+1}{2}$$

Proof. We calculate this directly from the symmetry in the binomial coefficients. Let $M=1,2,3,\dots$ and consider the sum

$$\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^{M+1}=\sum_{j=0}^{M}(-1)^j\binom{M}{M-j}j^{M+1}$$; let $k=M-j$

$$=\sum_{k=M}^{0}(-1)^{M-k}\binom{M}{k}(M-k)^{M+1}$$
$$=(-1)^{2M+1}\sum_{k=0}^{M}(-1)^k\binom{M}{k}(k-M)^{M+1}$$
$$=-\sum_{k=0}^{M}(-1)^k\binom{M}{k}(k-M)^{M+1}$$

where we reindexed, then factored out (-1) to arrive at the last equality. Now adding these expressions together and using Lemma 1 in [2] yields:

$$2\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^{M+1}$$
$$=\sum_{j=0}^{M}(-1)^j\binom{M}{j}[j^{M+1}-(j-m)^{m+1}]$$
$$=\sum_{j=0}^{M}(-1)^j\binom{M}{j}[(M+1)j^M M$$
$$-\binom{M+1}{2}j^{M-1}M^2+\cdots$$
$$=M(M+1)\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^M$$
$$-M^2\binom{M+1}{2}\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^{M-1}+\cdots$$
$$=2\binom{M+1}{2}\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^M-0+0-\cdots$$
$$=2\binom{M+1}{2}((-1)^M M!)$$
$$\therefore\sum_{j=0}^{M}(-1)^j\binom{M}{j}j^{M+1}=(-1)^M M!\binom{M+1}{2}$$

□

References

[1] Ledford, J. Approximating continuous functions with scattered translates of the Poisson kernel, *Missouri Journal of Mathematical Sciences.* **26** (2014), no. 1, 64–69

[2] J. Ledford, On the density of scattered translates of the general multiquadratic in C([a,b]). *New York J. Math.* **20** (2014), 145–151.

[3] M.J.D. Powell, Univariate multiquadric interpolation: Reproduction of linear polynomials, in *Multivariate Approximation and Interpolation (Duisberg 1989)*, Internat. Ser. Numer. Math. **94,** 227-240, Birkhäuser, Basel, 1990.