5-8-2020

Differential Invariants of Curves and Surfaces in Two and Three-Dimensional Geometries

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Abstract

We employ the Fels-Olver moving frame method to generate differential invariants of curves and surfaces in two and three-dimensional geometries.

Differential Invariants of Curves and Surfaces in Two and Three-Dimensional Geometries

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Acceptance date: May 8, 2020

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Chapter 1

Introduction

1.1 Historical

In Felix Klein's famed Erlangen Program he proposed that a geometry is determined by a collection of geometric transformations that act in a prescribed manner on a collection of points. The study of a geometric space is then carried out by studying the properties of the space and geometric objects in the space that remain invariant under the geometric transformations determining the geometry. While Klein's view on geometry is overarching and broad, it is familiar to almost everyone from their high school studies of mathematics and Euclidean geometry.

In the case of planar Euclidean geometry, the collection of points is agreed upon to be the points of a two-dimensional plane. The collection of geometric transformations determining Euclidean geometry consist of rotations and translations of the plane. Without specifying any other information, all of the hallmarks of Euclidean geometry can be deduced from its defining transformations. Studying Euclidean geometry from the point of Klein leads one to the basic invariants of Euclidean geometry: the distance between points and the angle between vectors (or line segments). Specifically, given a Euclidean transformation T and two points P and Q in the Euclidean plane, we can measure the distance $d(P,Q)$ between two points P and Q before the geometric transformation T is applied, or we can measure the distance $d(P',Q')$ between the transformed points $T(P) = P'$ and $T(Q) = Q'$ after the translation and we know that the distances will be identical. Likewise, we can measure the angle between two line segments $\overline{PQ}$ and $\overline{PR}$ determined by three distinct points P, Q, and R before a geometric transformation is applied or we can measure the angle between the line segments $\overline{P'Q'}$ and $\overline{P'R'}$ determined by the transformed points and we know that the measure of the indicated angles will be equal. It is in this sense that distance measure and angle measure are *invariants* of the geometric transformations that define Euclidean geometry.

In addition to being important in their own right, the invariants of a geometry can be used to study geometric objects determined by the underlying collection of points. Again calling on the standard study of Euclidean geometry from high school, the invariants of distance measure and angle measure allow us to completely classify triangles up to *congruence*. That is, we are able to use the invariants of Euclidean geometry to determine when two triangles $\triangle ABC$ and $\triangle PQR$ differ only by their location in the plane and not in a fundamental way. Said differently, we are able to use the invariants of Euclidean geometry to determine when there exists a geometric transformation of

Euclidean geometry that will carry triangle $\triangle ABC$ to triangle $\triangle PQR$, bringing point A to point P, point B to point Q and point C to point R. Examples of such conditions include the Side-Side-Side, Side-Angle-Side, Angle-Angle-Side, and Angle-Side-Angle conditions for the congruence of triangles. Note that this means that we are able to tell when two triangles $\triangle ABC$ and $\triangle PQR$ differ only by their location in space by measuring three invariant conditions of the triangle *without* needing to try all possible geometric transformations to determine whether there exists one which brings triangle $\triangle ABC$ to $\triangle PQR$.

The use of the invariants of a geometry to inform the study of the geometry extends beyond the study of geometric objects determined by a finite collection of points. In the study of planar Euclidean geometry, additional geometric of objects of interest are curves determined by a continuous collection of points. The arc length along a (directed) curve $\mathcal{C}$ between two points P and Q on the curve $\mathcal{C}$ is again an invariant, as is the arc length of the curve $\mathcal{C}$ itself. Additional invariants for (directed) curves are then found by checking how the direction of motion changes as one moves along the curve $\mathcal{C}$. As is the case for triangles in the Euclidean plane, the invariants of a curve $\mathcal{C}$ can be used to determine when two curves $\mathcal{C}$ and $\mathcal{C}'$ are congruent and differ only by their location in the plane. The details leading to this fundamental theorem are outlined below. Any standard textbook ([1],[13], [14]) the geometry of curves in the plane can be used as a reference.

Let $\mathcal{C}$ be a regular directed curve $\gamma(s) = (x(s), y(s))$ parametrized by arc length. The tangent vector $\vec{t}(s) = \frac{d\gamma}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ always points in the direction of motion and is of unit length when the curve $\mathcal{C}$ is parametrized by arc length. There is a well-defined normal vector $\vec{n}(s)$ along the curve $\mathcal{C}$ that is a $90°$ rotation of $\vec{t}(s)$ in the counterclockwise direction. The pair of vectors $\vec{t}(s)$ and $\vec{n}(s)$ determine a *moving frame* along the curve $\mathcal{C}$ and the curve $\mathcal{C}$ is determined by how the instantaneous direction of motion $\vec{t}(s)$ of $\mathcal{C}$ changes. It can be shown [1] that instantaneous rate of change $\frac{d\vec{t}}{ds}$ of the unit tangent vector $\vec{t}(s)$ is always perpendicular to $\vec{t}(s)$ and must be a multiple of the normal vector $n\vec{(s)}$. This allows us to conclude that there exists a scalar function $\kappa(s)$ called the *curvature of* $\mathcal{C}$ such that

$$
\begin{aligned}
\frac{d\vec{t}}{ds} &= \kappa(s)\vec{n}(s) \\
\frac{d\vec{n}}{ds} &= -\kappa(s)\vec{t}(s).
\end{aligned}
$$

As the name suggests, the curvature function $\kappa(s)$ is a measure of how the curve $\mathcal{C}$ curves in the plane, providing a measure of how the instantaneous direction of motion $\vec{t}(s)$ of the curve $\mathcal{C}$ changes as one moves along $\mathcal{C}$. See Figure 1.1 for a visual of the unit tangent and normal vector as it moves along a curve as a moving frame.

The highlight of this construction is the following theorem.

Theorem 1.1.1. *Given a differentiable function $\kappa(s)$, there exists exactly one curve $\mathcal{C}$, determined up to positioning in the the plane, parametrized by arc length with curvature $\kappa(s)$.*

It is in this sense that the curvature $\kappa(s)$ of a regular curve $\mathcal{C}$ parametrized by arc length in the Euclidean plane determines the curve $\mathcal{C}$. The use of the moving frame outlined above for curves

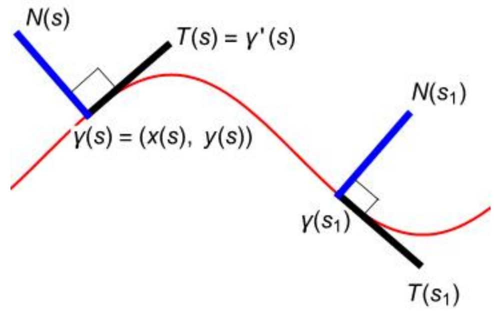

Figure 1.1: A visualization of the moving frame along a curve $\mathcal{C}$ in the Euclidean plane.

in the Euclidean plane more or less completes the study of the geometry of the Euclidean plane in the vision of Klein, as invariants of both the geometry and its geometric objects are completely understood.

1.2 Outline of Thesis

In this thesis we use the moving frame method by developed Peter Olver and Mark Fels [2] to investigate two and three-dimensional geometries in the spirit of Klein, identifying differential invariants of curves and surfaces (geometric objects) of the geometries in question.

In [2], Mark Fels and Peter Olver outlined a method of moving frames that can be used to catalog and classify invariants of geometric objects in a geometric space X that is determined by a collection of geometric transformations $\mathcal{G}$ on X. In comparison to the moving frame method for curves in the Euclidean plane outlined above, the Fels-Olver moving frame method does not rely on advance knowledge of the geometric space being investigated and can instead be used as way to investigate the geometry of the space itself. The Fels-Olver moving frame method can easily be adapted to investigate invariants of geometric objects of different dimensions that reside in the same underlying geometric space. For example, in a three-dimensional geometric space, there are one-dimensional geometric objects (curves) and two-dimensional geometric objects (surfaces) that can be investigated. The ability to investigate both the curves and surfaces of a geometric space on the same conceptual and analytic foundation is a significant benefit of the Fels-Olver moving method in comparison to traditional moving methods [3], [4], [5], [15].

We complete an investigation of three distinct geometries using the Fels-Olver moving method. In all cases, the point set of the geometric space will be either the ordinary two-dimensional Cartesian plane $\mathbb{R}^2$ or the ordinary three-dimensional Cartesian space $\mathbb{R}^3$. The collection of geometric transformations acting on the points of the space will determine the geometry under investigation. We provide a classification of the invariants of curves (in the case of a two-dimensional geometry) or a classification of the invariants of curves and surfaces (in the case of a three-dimensional geometry).

The three geometric spaces that we investigate are outlined below.

1. The Lorentz-Minkowski plane. We take as a point set $\mathbb{R}^2$ and identify points P in coordinates as $P(x,y)$. The geometric transformations defining the geometry of the Lorentz-Minkowski plane are identified with three parameters (a,b,θ) and transform the points of the plane according to the transformation laws[1]

$$\begin{aligned}\bar{x} &= x\cosh\theta + y\sinh\theta + a\\ \bar{y} &= x\sinh\theta + y\cosh\theta + b.\end{aligned}$$

   We then use the Fels-Olver moving frame method to find the properties of a generic curve $\mathcal{C}$ that remain unchanged when transformed under the indicated geometric transformations.

2. The Heisenberg Group $\mathcal{H}^3_R$. We take as a point set $\mathbb{R}^3$ and identify points P in coordinates as $P(x,y,z)$. The geometric transformations defining the geometry of $\mathcal{H}^3_R$ are identified with four parameters (a,b,c,θ) and transform the points of the $\mathbb{R}^3$ according to the transformation laws

$$\begin{aligned}\bar{x} &= x\cos\theta - y\sin\theta + a\\ \bar{y} &= x\sin\theta + y\cos\theta + b\\ \bar{z} &= z + c + \frac{1}{2}\left(a\sin\theta - b\cos\theta\right)x + \frac{1}{2}\left(a\cos\theta + b\sin\theta\right)y.\end{aligned}$$

   We then use the Fels-Olver moving frame method to find the properties of generic curves $\mathcal{C}$ and generic surfaces $\mathcal{S}$ that remain unchanged when transformed under the indicated geometric transformations.

3. The Heisenberg Group $\mathcal{H}^3_L$. We take as a point set $\mathbb{R}^3$ and identify points P in coordinates as $P(x,y,z)$. The geometric transformations defining the geometry of $\mathcal{H}^3_R$ are identified with four parameters (a,b,c,θ) and transform the points of the $\mathbb{R}^3$ according to the transformation laws

$$\begin{aligned}\bar{x} &= x\cosh\theta + y\sinh\theta + a\\ \bar{y} &= x\sinh\theta + y\cosh\theta + b\\ \bar{z} &= z + c + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right)x + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y.\end{aligned}$$

   We then use the Fels-Olver moving frame method to find the properties of generic curves $\mathcal{C}$ and generic surfaces $\mathcal{S}$ that remain unchanged when transformed under the indicated geometric transformations.

[1] Note for the purposes of comparison that the geometry of the Euclidean plane can also be investigated using the Fels-Olver moving frame method. The transformation laws encoding the Euclidean geometry of rotations and translations are given by $\bar{x} = x\cos\theta - y\sin\theta + a$ and $\bar{y} = x\sin\theta + y\cos\theta + b$. We will review this in Chapter 3 as a way to demonstrate the Fels-Olver moving frame method.

The investigation of all four geometries will follow the Fels-Olver moving frame method and require that the geometric transformations are prolonged to the jet space of curves/surfaces in the geometric space under investigation. Extensive use will be made throughout the investigation of the geometry of both derivatives and partial derivatives, including tangent lines to curves and tangent planes to surfaces. See [9] and [10] for details.

Chapter 2

Technical Material

In this chapter we outline the technical material needed to communicate the results and employ the Fels-Olver moving frame method. For additional details related to groups and group actions one can consult a standard modern algebra textbook, but we follow [11]. For additional details and a complete generalization of jet bundles for curves and surfaces, see [8], [9], [10], [11]. For additional details on the prolongation of a group action to the jet bundle, see any of the textbooks [8], [9], [10], [11]. An adequate and thorough summary of all of this material in the setting of moving frames can be found in [2].

2.1 Groups and Group Actions

The foundation of Klein's view on geometry rests on groups and group actions on a space. A group is central to Klein's view on geometry as it provides a notion of symmetry. Recall the definition of a group $\mathcal{G}$.

Definition 1 (Group)**.** *A group $\mathcal{G}$ is defined as a set of elements with a binary operation that satisfies the properties of closure, associativity, the identity property, and the inverse property.*

Remark 2.1.1. *We tend to represent the binary operation of a group $\mathcal{G}$ using multiplicative notation $g_1 g_2$ for $g_1, g_2 \in \mathcal{G}$. Often our groups $\mathcal{G}$ will be groups of matrices and the binary operation is ordinary matrix multiplication.*

Example 2.1.1 (Rotations of $\mathbb{R}^2$)**.** *The set of rotations of the plane that fix the origin is realized as the set*

$$\mathcal{G} = \left\{ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\},$$

and becomes a group with matrix multiplication defining the binary (or group) operation. It can be explicitly checked that the required properties of a group are satisfied. Intuitively, a rotation R_{θ_1} followed by a second rotation R_{θ_2} will produce a third rotation $R_{\theta_3} = R_{\theta_1+\theta_2}$; rotations are associative; the rotation of by zero degrees will act as the identity; and the inverse of a rotation R_θ will by a rotation $R_{-\theta}$ of equal magnitude but in the opposite direction.

Definition 2 (Group Action)**.** *Let $\mathcal{G}$ be a group and X a set. A $\mathcal{G}$-action on X is a function $\mu : \mathcal{G} \times X \to X$, typically written $\mu(g, x) = g \cdot x$, that satisfies*

1. *$\mu(Id, x) = Id \cdot x = x$ for all $x \in \mathcal{G}$, and*
2. *$\mu(g_1, \mu(g_2, x)) = \mu(g_1 g_2, x)$, or equivalently $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, for all $g_1, g_2 \in \mathcal{G}$ and all $x \in X$.*

Remark 2.1.2. *A group action of a group $\mathcal{G}$ on a set X is a way to realize $\mathcal{G}$ as a set of transformations on the set X.*

Remark 2.1.3. *For our purposes, we will always begin with a group action on $\mathbb{R}^2$ or $\mathbb{R}^3$.*

Remark 2.1.4. *In much of this work, the group $\mathcal{G}$ under consideration will be a group of matrices and we will have a $\mathcal{G}$-action on either $X = \mathbb{R}^2$ or $X = \mathbb{R}^3$ defined by matrix multiplication.*

Example 2.1.2. *Let $\mathcal{G} = \left\{ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$ be the group of rotations on the Cartesian plane $X = \mathbb{R}^2$ and let $p = \begin{pmatrix} x \\ y \end{pmatrix}$ be a point in $\mathbb{R}^2$ represented as a column vector. Then there is $\mathcal{G}$-action on $\mathbb{R}^2$ defined by matrix multiplication:*

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}.$$

From, this, we get that

$$\bar{x} = x\cos\theta - y\sin\theta$$
$$\bar{y} = x\sin\theta + y\cos\theta.$$

It is in this way that we think of a $\mathcal{G}$-action as a set of transformations $(x, y) \mapsto g \cdot (x, y) = (\bar{x}, \bar{y})$ on X.

A group action on a set X can satisfy additional properties, but we focus on one here.

Definition 3 (Transitive)**.** *Let $\mathcal{G}$ be a group and X be a set. A $\mathcal{G}$-action on X is said to be* transitive *if for all $x_1, x_2 \in X$, there exists $g \in \mathcal{G}$ such that $g \cdot x_1 = x_2$.*

Remark 2.1.5. *A $\mathcal{G}$-action on X is transitive if any point in X can be brought to any other point in X by an element of $\mathcal{G}$*

We now define a group of *geometric transformations* on a space $X = \mathbb{R}^p$, $p = 2$ or $p = 3$. For this we will require that our group $\mathcal{G}$ be a matrix Lie group. We will not pursue the abstract study or properties of matrix Lie groups in the thesis and will settle on giving an indication of what we are requiring for a group of geometric transformations.

Definition 4 (Matrix Lie group)**.** *A group $\mathcal{G}$ is said to be a* matrix Lie group *if $\mathcal{G}$ can be a realized as a closed subgroup of $GL_n(\mathbb{R})$, the group of $n \times n$ invertible matrices with entries in $\mathbb{R}$, and whose elements depend continuously on the parameters.*

Definition 5 (Group of Geometric Transformations)**.** *A group $\mathcal{G}$ is a* group of geometric transformations *on $X = \mathbb{R}^p$, $p = 2$ or $p = 3$, if*

1. *$\mathcal{G}$ is a matrix Lie group, and*
2. *there is a transitive $\mathcal{G}$-action on X such that the components of the action $\mu : \mathcal{G} \times X \to X$ are differentiable of all orders.*

Example 2.1.3. *Let $\mathcal{G} = \left\{ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$ be the group of rotations on the Cartesian plane $X = \mathbb{R}^2$. The group $\mathcal{G}$ is matrix Lie group but it is not a group of geometric transformations on X. The $\mathcal{G}$-action on $X = \mathbb{R}^2$ is not transitive as a point $p = \begin{pmatrix} x \\ y \end{pmatrix}$ can only be rotated around a circle of a fixed radius $r = \sqrt{x^2 + y^2}$ by elements of $\mathcal{G}$.*

We can now think of a group of geometric transformations on $X = \mathbb{R}^2$ or $\mathbb{R}^3$ as defining a geometry on X.

We close the section with the definition of an invariant for a $\mathcal{G}$-action. Informally, an invariant for a $\mathcal{G}$-action is a quantity that does not change under the action of $\mathcal{G}$.

Definition 6 (Invariant)**.** *An* invariant *for a $\mathcal{G}$-action on a set X is a function $f : X \to \mathbb{R}$ such that $f(g \cdot x) = f(x)$ for all $g \in \mathcal{G}$ and all $x \in X$.*

Remark 2.1.6. *Note that if a $\mathcal{G}$-action on a set X is transitive, then any invariant is a constant function. If x_1 and x_2 are elements of X and the $\mathcal{G}$-action is transitive, then there exists $g \in \mathcal{G}$ such that $g \cdot x_1 = x_2$. But if f is an invariant, then $f(x_1) = f(g \cdot x_1) = f(x_2)$ and this must hold for all x_1 and x_2 in X.*

2.2 Prolonged Actions and Jet Spaces

We now establish what is needed to find differential invariants of curves and surfaces under a group of geometric transformations $\mathcal{G}$ on $\mathbb{R}^2$ or $\mathbb{R}^3$. Everything extends to higher dimensional spaces with the appropriate changes. For full details, see [2], [8], [9], [10], or [11].

To setup the general case, we first consider the case of curves in the Cartesian plane $\mathbb{R}^2$. We will always work locally and assume that a curve $\mathcal{C}$ is given by the graph of a function $y = y(x)$. Since we are working locally, we will ignore issues related to the domain of the function and generally assume that the domain is all of $\mathbb{R}$. The image of the curve $\mathcal{C}$ is the set of points:

$$\mathcal{C} = \left\{ (x, y(x)) \Big| x \in \mathbb{R} \right\}.$$

The $\mathcal{G}$-action on $\mathbb{R}^2$ extends to curves and gives rise to an induced action. If $g \in \mathcal{G}$, then g transforms a curve $\mathcal{C}$ by

$$g \cdot \mathcal{C} = \left\{ g \cdot (x, y(x)) \Big| x \in \mathbb{R} \right\} = \overline{\mathcal{C}}.$$

Our primary concern is to find invariants of the curve $\mathcal{C}$, or properties of $\mathcal{C}$ that do not change under the $\mathcal{G}$-action. According to Remark 2.1.6, the only invariants for the action of a group of geometric transformations on $\mathbb{R}^2$ are constant functions, as any point $\mathbb{R}^2$ can be brought to any other point in $\mathbb{R}^2$ by an element of $\mathcal{G}$. As a result there can be no meaningful invariants of a curve $\mathcal{C}$ based solely on the points of $\mathcal{C}$. Any meaningful invariant of $\mathcal{C}$ must depend on more than the points of $\mathcal{C}$. To identify meaningful invariants of $\mathcal{C}$, we need to *prolong* the action of $\mathcal{G}$ to the derivatives and infinitesimal approximations of $\mathcal{C}$.

When $g \in \mathcal{G}$ acts on $\mathbb{R}^2$ and transforms a curve $\mathcal{C}$, there is also an induced (geometric) action on the tangent lines of $\mathcal{C}$. Looking at a point $(x, y(x))$ on $\mathcal{C}$, the transformation $g \in \mathcal{G}$ will not only take the point $(x, y(x))$ to a point $(\overline{x}, \overline{y}\,(\overline{x}))$, but it will take the tangent line l to $\mathcal{C}$ at $(x, y(x))$ to the tangent line $\bar{l}$ of $\overline{\mathcal{C}}$ at the point $(\overline{x}, \overline{y}(\overline{x}))$. Since a tangent line l is determined by the point $(x, y(x))$ and the derivative $y'(x)$, a prolonged action of $\mathcal{G}$ on derivatives is defined by requiring that it satisfy

$$g \cdot l = \bar{l}.$$

The $\mathcal{G}$-action can then be extended to derivatives of any order by requiring it to take n^{th} order approximating objects of $\mathcal{C}$ at $(x, y(x))$ defined by the Taylor series to n^{th} order approximating objects of $\overline{\mathcal{C}}$ at $((\overline{x}, \overline{y}\,(\overline{x}))$. See [11] for full details. We outline what is required for curves in $\mathbb{R}^2$, curves in $\mathbb{R}^3$, and surfaces in $\mathbb{R}^3$ below.

2.2.1 Prolonged $\mathcal{G}$-Actions on Curves in $\mathbb{R}^2$

For this section we closely follow [11]. Let $\mathcal{G}$ be a group of geometric transformations on $\mathbb{R}^2$. We will denote points in $\mathbb{R}^2$ in coordinate form by (x, y) and we will express the $\mathcal{G}$-action on $\mathbb{R}^2$ by

$$g \cdot (x, y) = (\overline{x}(x, y, g), \overline{y}(x, y, g)).$$

We are suppressing the transformation parameters of $\mathcal{G}$ but indicating that the transformed point $(\overline{x}, \overline{y})$ depends on x, y, and the group transformation g. The transformation laws are then

$$\overline{x} = \overline{x}(x, y, g) \tag{2.1}$$
$$\overline{y} = \overline{y}(x, y, g). \tag{2.2}$$

To find the prolonged action of $\mathcal{G}$ on curves in $\mathbb{R}^2$ we work locally and consider curves $\mathcal{C}$ which can be described by the graph of a function $y = y(x)$. We use the $(n+2)$-dimensional Cartesian space $\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R})$ with suggestively named coordinates to collect the n-jets of curves:

$$\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R}) = \left\{ \left(x, y, y', y'', \ldots, y^{(n)} \right) \right\} \sim \mathbb{R}^{n+2}.$$

Definition 7 (n-jet)**.** *The* n-jet *of a curve* $\mathcal{C} = \left\{ (x, y(x)) \,\middle|\, x \in \mathbb{R} \right\}$ *is the curve*

$$\mathrm{j}^{(n)}(\mathcal{C}) = \left\{ \left(x, y(x), y'(x), y''(x), \ldots y^{(n)}(x) \right) \,\middle|\, x \in \mathbb{R} \right\} \subset \mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R}).$$

Example 2.2.1. *Let $\mathcal{C}$ be the curve in $\mathbb{R}^2$ given by $(x, \cos x + x^4)$. The three-jet of $\mathcal{C}$ is the curve*

$$\mathrm{j}^{(3)}(\mathcal{C}) = \left\{ (x, \cos x + x^4, -\sin x + 4x^3, -\cos x + 12x^2, \sin x + 24x) \,\middle|\, x \in \mathbb{R} \right\} \subset \mathcal{J}^{(3)}(\mathbb{R}, \mathbb{R}).$$

Remark 2.2.1. *Not every curve in $\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R})$ is the n-jet of a curve $\mathcal{C}$ in $\mathbb{R}^2$. To be the n-jet of a curve in $\mathbb{R}^2$, there is a compatibility condition that must be satisfied by the components of the curve: each component after the second component must be the derivative of the component that immediately precedes it.*

The $\mathcal{G}$-action on $\mathbb{R}^2$ prolonged to curves and derivatives of curves is represented by an action of $\mathcal{G}$ on $\mathcal{J}^{(1)}(\mathbb{R}, \mathbb{R})$ and follows from the chain rule. We first outline the construction for prolonging the action of $\mathcal{G}$ to $\mathcal{J}^{(1)}(\mathbb{R}, \mathbb{R})$. The $\mathcal{G}$-action is prolonged to the higher order jets $\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R})$ by applying a *prolonged differential operator*.

We begin by noting that while a curve $\mathcal{C}$ starts out as the graph of a function $y = y(x)$, the transformed curve

$$g \cdot \mathcal{C} = g \cdot \left\{ (x, y(x)) \,\middle|\, x \in \mathbb{R} \right\} = \overline{\mathcal{C}}$$

does not have to be the graph of a function of x. The transformed curve will (at least locally) be described by a function where $\overline{y}$ is a function of the independent variable $\overline{x}$: $\overline{y} = \overline{y}(\overline{x})$. We can then implicitly differentiate to calculate $\frac{d\overline{y}}{d\overline{x}}$. We have the following definition.

Definition 8. *Let $\mathcal{G}$ be a group of geometric transformations on $\mathbb{R}^2$. The prolonged action of $\mathcal{G}$ to the one-jets of curves $\mathcal{C}$ that are described by the graph of a function $y = y(x)$ is defined by*

$$g \cdot \frac{dy}{dx} = \frac{d\overline{y}}{d\overline{x}}.$$

Implicit differentiation of (2.1) and (2.2) gives

$$\frac{d\overline{y}}{d\overline{x}} = \frac{\frac{\partial \overline{y}}{\partial x}}{\frac{\partial \overline{x}}{\partial x}} = \frac{\frac{\partial \overline{y}}{\partial x} + \frac{\partial \overline{y}}{\partial y}\frac{dy}{dx}}{\frac{\partial \overline{x}}{\partial x} + \frac{\partial \overline{x}}{\partial y}\frac{dy}{dx}}. \tag{2.3}$$

To simplify notation in the above, we make use of the *implicit differentiation operator*.

Definition 9 (The Implicit Differentiation Operator)**.** *The* implicit differentiation operator *for curves in $\mathbb{R}^2$ that are described by the graph of a function $y = y(x)$ is*

$$D_x = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \ldots \tag{2.4}$$

Remark 2.2.2. *When applying the implicit differentiation operator we treat all appearances of the dependent variable y and its derivatives as free variables.*

The prolonged transformation of the $\mathcal{G}$-action in (2.3) can then be expressed as

$$\frac{d\overline{y}}{d\overline{x}} = \frac{D_x\overline{y}}{D_x\overline{x}} = \overline{y}',$$

where $\overline{y}$ and $\overline{x}$ are given by (2.1) and (2.2).

The prolonged action of $\mathcal{G}$ to the space of n-jets is then defined inductively by

$$\overline{y}^{(n)} = g \cdot y^{(n)} = \frac{1}{D_x\overline{x}} D_x\left(\overline{y}^{(n-1)}\right), \qquad \text{for all } g \in \mathcal{G}, n \geq 1.$$

The differentiation operator $\mathcal{D}_x = \frac{1}{D_x\overline{x}} D_x$ is important enough to give it a name.

Definition 10 (Prolonged Transformation Operator)**.** *Let $\mathcal{G}$ be a group of geometric transformations on $\mathbb{R}^2$ with $\mathcal{G}$-action described by (2.1) and (2.2). The differential operator $\mathcal{D}_x = \frac{1}{D_x\overline{x}} D_x$ is called the* prolonged transformation operator *for the $\mathcal{G}$-action on $\mathbb{R}^2$ to the action of $\mathcal{G}$ on $\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R})$.*

2.2.2 Prolonged $\mathcal{G}$-actions on Curves in $\mathbb{R}^3$

Let $\mathcal{G}$ be a group of geometric transformations acting on $\mathbb{R}^3$. The prolonged $\mathcal{G}$-action on curves in $\mathbb{R}^3$ follows directly from the above once one accounts for the fact that additional variables will now depend on the independent variable. For reference we outline the relevant definitions for the prolongation of the $\mathcal{G}$-action to curves $\mathcal{C}$ in $\mathbb{R}^3$. We will assume that we are working with curves $\mathcal{C}$ where both y and z are functions of the independent variable x:

$$\mathcal{C} = \left\{(x, y(x), z(x)) \in \mathbb{R}^3 \middle| x \in \mathbb{R}\right\}.$$

As before we assume for simplicity that the domain is all of $\mathbb{R}$.

We will denote points in $\mathbb{R}^3$ in coordinate form by (x, y, z) and we will express the $\mathcal{G}$-action on $\mathbb{R}^3$ by

$$g \cdot (x, y, z) = (\overline{x}(x, y, z, g), \overline{y}(x, y, z, g), \overline{z}(x, y, z, g)).$$

As before we are suppressing the transformation parameters of $\mathcal{G}$ but indicating that the transformed point $(\overline{x}, \overline{y}, \overline{z})$ depends on x, y, z and the group transformation g. The transformation laws are then

$$\overline{x} = \overline{x}(x, y, z, g) \tag{2.5}$$

$$\overline{y} = \overline{y}(x, y, z, g) \tag{2.6}$$

$$\overline{z} = \overline{z}(x, y, z, g) \tag{2.7}$$

We introduce the $(3 + 2n)$-dimensional Cartesian space $\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R}^2)$ with suggestively named coordinates for the n-jets of curves $\mathcal{C} = \left\{(x, y(x), z(x)) \middle| x \in \mathbb{R}\right\}$:

$$\mathcal{J}^{(n)}(\mathbb{R}, \mathbb{R}^2) = \left\{\left(x, y, z, y', z', y'', z'', \ldots, y^{(n)}, z^{(n)}\right)\right\} \sim \mathbb{R}^{3+2n}.$$

Definition 11 (n-jet)**.** *The n-jet of a curve $\mathcal{C} = \left\{ (x, y(x), z(x)) \middle| x \in \mathbb{R} \right\} \subset \mathbb{R}^3$ is the curve*

$$\mathrm{j}^{(n)}(\mathcal{C}) = \left\{ \left(x, y(x), z(x), y'(x), z'x, y''(x), z''(x), \ldots, \ldots y^{(n)}(x), z^{(n)}(x) \right) \middle| x \in \mathbb{R} \right\} \subset \mathcal{J}^{(n)}\left(\mathbb{R}, \mathbb{R}^2\right).$$

Definition 12 (The Implicit Differentiation Operator)**.** *The* implicit differentiation operator *for curves in $\mathbb{R}^3$ where y and z are functions of the independent variable is*

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} + y'' \frac{\partial}{\partial y'} + z'' \frac{\partial}{\partial z'} \ldots$$

The prolonged transformation of the $\mathcal{G}$-action to $\mathcal{J}^{(1)}\left(\mathbb{R}, \mathbb{R}^2\right)$ is then defined by

$$\frac{d\overline{y}}{d\overline{x}} = \frac{D_x \overline{y}}{D_x \overline{x}} = \overline{y}' \tag{2.8}$$

and

$$\frac{d\overline{z}}{d\overline{x}} = \frac{D_x \overline{z}}{D_x \overline{x}} = \overline{z}', \tag{2.9}$$

where $\overline{y}$ and $\overline{x}$ are given by (2.5) and (2.6).

As in the case of $\mathbb{R}^2$, the prolonged transformation of an element $g \in \mathcal{G}$ on $\mathbb{R}^3$ to $\mathcal{J}^{(n)}\left(\mathbb{R}, \mathbb{R}^2\right)$ is defined inductively by repeated application of the differential operator $\mathcal{D}_x = \frac{1}{D_x \overline{x}} D_x$. Specifically

$$\overline{y}^{(n)} = g \cdot y^{(n)} = \frac{1}{D_x \overline{x}} D_x \overline{y}^{(n-1)} = \mathcal{D}_x \overline{y}^{(n-1)} \tag{2.10}$$

and

$$\overline{z}^{(n)} = g \cdot z^{(n)} = \frac{1}{D_x \overline{x}} D_x \overline{z}^{(n-1)} = \mathcal{D}_x \overline{z}^{(n-1)} \tag{2.11}$$

We summarize this with the following definition

Definition 13 (Prolonged Transformation Operator)**.** *Let $\mathcal{G}$ be a group of geometric transformations on $\mathbb{R}^3$ with $\mathcal{G}$-action described by (2.5), (2.6), and (2.7). Let D_x be the implicit differentiation operator as in Definition 12. The differential operator $\mathcal{D}_x = \frac{1}{D_x \overline{x}} D_x$, is called the* prolonged transformation operator *for the $\mathcal{G}$-action on $\mathbb{R}^3$ to the action of $\mathcal{G}$ on $\mathcal{J}^{(n)}\left(\mathbb{R}, \mathbb{R}^2\right)$.*

2.2.3 Prolonged $\mathcal{G}$-actions on Surfaces in $\mathbb{R}^3$

Like curves, when a group of geometric transformations $\mathcal{G}$ acts on $\mathbb{R}^3$, there is an induced action on surfaces $\mathcal{S}$ in $\mathbb{R}^3$. The idea of the prolonged action of $\mathcal{G}$ to surfaces is the same. If $\mathcal{S}$ is a surface in $\mathbb{R}^3$, then a transformation of $\mathbb{R}^3$ by $g \in \mathcal{G}$ will induce a transformation of $\mathcal{S}$ that we prolong geometrically. For example if g takes $\mathcal{S}$ to $g \cdot \mathcal{S} = \overline{\mathcal{S}}$, then g will also take the tangent plane of $\mathcal{S}$ to $\overline{\mathcal{S}}$. We interpret this as an action of $\mathcal{G}$ on first order partial derivatives. We can then prolong the action higher order derivatives. The main difference is that a surface depends on two independent variables, not one. To account for the difference, we need an additional prolonged transformation operator. We will record the relevant definitions following [2] and [11].

We will typically work with surfaces $\mathcal{S}$ that can be described as the graph of a function $z = z(x, y)$:

$$\mathcal{S} = \left\{ (x, y, z(x, y)) \in \mathbb{R}^3 \middle| (x, y) \in \mathbb{R}^2 \right\}.$$

For simplicity we will assume that the domain of the function $z = z(x, y)$ is all $(x, y) \in \mathbb{R}^2$. We will use the notation of (2.5), (2.6), (2.7) to represent the transformation $g \cdot (x, y, z) = (\overline{x}, \overline{y}, \overline{z})$ of a point (x, y, z) by $g \in \mathcal{G}$.

The n-jets of surfaces will be recorded in the Cartesian space with the indicated coordinates

$$\mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}^1\right) = \{(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}, \ldots, z_K)\},$$

where N ranges over all partial derivative strings of order less or equal to n. N then represents symmetric (mixed-partial derivatives commute) strings of xs and ys of length less than or equal n . We will order our partial derivatives of given degree using the dictionary order.

Definition 14 (n-jet)**.** *The* n-jet *of a surface* $\mathcal{S} = \left\{(x, y, z\,(z, y)) \,\middle|\, (x, y) \in \mathbb{R}^2\right\} \subset \mathbb{R}^3$ *is the surface*

$$\mathrm{j}^{(n)}\left(\mathcal{S}\right) = \left\{(x, y, z(x, y), z_x(x, y), z_y(x, y), z_{xx}(x, y), z_{xy}(x, y), z_{yy}(x, y), \ldots, z_K(x, y)) \,\middle|\, (x, y) \in \mathbb{R}^2\right\}.$$

The n-jet $\mathrm{j}^{(n)}\left(\mathcal{S}\right)$ *is a subset of* $\mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}^1\right)$.

Example 2.2.2. *Let* $\mathcal{S} = \left\{\left(x, y, xy^2 + y\cos x\right)\right\}$. *The two-jet of* $\mathcal{S}$ *is*

$$\mathrm{j}^{(2)}\left(\mathcal{S}\right) = \left\{\left(x, y, xy^2 + y\cos x, y^2 - y\sin x, 2xy + \cos x, -y\cos x, 2y - \sin x, 2x\right) \,\middle|\, (x, y) \in \mathbb{R}^2\right\}.$$

As with our discussion of curves in $\mathbb{R}^2$, if a surface $\mathcal{S}$ is the graph of a function $z = z(x, y)$, then a transformed surface $\overline{\mathcal{S}} = g \cdot \mathcal{S}$ does not have to continue to be the graph of a function of the independent variables x and y. The surface $\overline{\mathcal{S}}$ will be the graph of a function $\overline{z} = \overline{z}\left(\overline{x}, \overline{y}\right)$, where $\overline{x}$ and $\overline{y}$ the independent variables. Determining the prolonged transformations of $\overline{z}_{\overline{x}}$ and $\overline{z}_{\overline{y}}$ requires the use of implicit differentiation operators.

Definition 15 (The Implicit Differentiation Operators)**.** *The* implicit differentiation operators *for surfaces in* $\mathbb{R}^3$ *where* $z = z(x, y)$ *is a function of the independent variables* x *and* y *are*

$$D_x = \frac{\partial}{\partial x} + z_x\frac{\partial}{\partial z} + z_{xx}\frac{\partial}{\partial z_x} + z_{xy}\frac{\partial}{\partial z_y} + z_{xxx}\frac{\partial}{\partial z_{xx}} + z_{xxy}\frac{\partial}{\partial z_{xy}}\ldots$$

and

$$D_y = \frac{\partial}{\partial y} + z_y\frac{\partial}{\partial z} + z_{xy}\frac{\partial}{\partial z_x} + z_{yy}\frac{\partial}{\partial z_y} + z_{xxy}\frac{\partial}{\partial z_{xx}} + z_{xyy}\frac{\partial}{\partial z_{xy}}\ldots$$

The prolonged action of $\mathcal{G}$ to surfaces in $\mathbb{R}^3$ that are described as the graph of a function $z = z(x, y)$ (or the prolonged action of $\mathcal{G}$ to $\mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}^1\right)$) is given by repeated application of prolonged transformation operators. We take the following as a definition. See [2], [8], or [10] for complete details and justification.

Definition 16 (Prolonged Transformation Operators)**.** *Let* $\mathcal{G}$ *be a group of geometric transformations on* $\mathbb{R}^3$ *with* $\mathcal{G}$*-action described by* (2.5)*,* (2.6)*, and* (2.7)*. Let* D_x *and* D_y *be the implicit differentiations operators as in Definition 15. Let*

$$\mathbb{J} = \begin{pmatrix} D_x\overline{x} & D_y\overline{x} \\ D_x\overline{y} & D_y\overline{y} \end{pmatrix}.$$

The differential operators $\mathcal{D}_x$ *and* $\mathcal{D}_y$ *defined by*

$$\begin{pmatrix} \mathcal{D}_x \\ \\ \mathcal{D}_y \end{pmatrix} = \mathbb{J}^{-T} \begin{pmatrix} D_x \\ \\ D_y \end{pmatrix} = \frac{1}{\det \mathbb{J}} \begin{pmatrix} D_y\overline{y} & -D_x\overline{y} \\ \\ -D_y\overline{x} & D_x\overline{x} \end{pmatrix} \begin{pmatrix} D_x \\ \\ D_y \end{pmatrix}$$

are called the prolonged transformation operators *for the* $\mathcal{G}$*-action on* $\mathbb{R}^3$ *to the action of* $\mathcal{G}$ *on* $\mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}^1\right)$.

Carrying out the matrix multiplication in the above definition, the prolonged transformation operators are expressed as

$$\mathcal{D}_x = \frac{1}{D_x\overline{x}\, D_y\overline{y} - D_y\overline{x}\, D_x\overline{y}} \left(D_y\overline{y}\, D_x - D_x\overline{y}\, D_y\right) \tag{2.12}$$

and

$$\mathcal{D}_y = \frac{1}{D_x\overline{x}\, D_y\overline{y} - D_y\overline{x}\, D_x\overline{y}} \left(-D_y\overline{x}\, D_x + D_x\overline{x}\, D_y\right) \tag{2.13}$$

The formula for the prolonged action of $\mathcal{G}$ to $\mathcal{J}^{(1)}\left(\mathbb{R}^2, \mathbb{R}\right)$ are

$$\overline{z}_{\overline{x}} = g \cdot z_x = \mathcal{D}_x\overline{z} \qquad \textit{and} \qquad \overline{z}_{\overline{y}} = g \cdot z_y = \mathcal{D}_y\overline{z}.$$

The formula for the higher order prolongations of $\mathcal{G}$ to $\mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}\right)$ are defined inductively by

$$\overline{z}_{K,\overline{x}} = \mathcal{D}_x\left(\overline{z}_K\right) \qquad \textit{and} \qquad \overline{z}_{K,\overline{y}} = \mathcal{D}_y\left(\overline{z}_K\right),$$

where K is a partial derivative subscript string of degree $n-1$, $n \geq 1$.

2.3 Fels-Olver Method of Moving Frames

In [2] , Mark Fels and Peter Olver outlined a method of moving frames that can be used to catalog and classify invariants of geometric objects in a geometric space X that is determined by a collection of geometric transformations $\mathcal{G}$ on X. In comparison to the moving frame method outlined above for curves in the Euclidean plane, the Fels-Olver moving frame method does not rely on advance knowledge of the geometric space being investigated and can instead be used as way to investigate the geometry of the space itself.

Further, the Fels-Olver moving frame method can be easily adapted to investigate invariants of geometric objects of different dimensions that reside in the same underlying geometric space.

For example, in a three-dimensional geometric space, there are one-dimensional geometric objects (curves) and two-dimensional geometric objects (surfaces) that can be investigated.

The ability to investigate both the curves and surfaces of a geometric space on the same conceptual and analytic foundation is a significant benefit of the Fels-Olver moving method in comparison to traditional moving methods [3], [4], [5], [15].

2.3.1 Moving Frame Maps

We now outline the relevant definitions and the algorithm for the construction of a Fels-Olver moving frame.

Notation 2.3.1. *For a group $\mathcal{G}$ acting on a set X, we will denote the action of an element $g \in \mathcal{G}$ on $x \in X$ by $g \cdot x \in X$. The group operation of $\mathcal{G}$ will be denoted by ordinary multiplication.*

Definition 17 (Moving Frame Map)**.** *A* moving frame map *for the action of a group $\mathcal{G}$ on a set X is a map $\rho : X \to \mathcal{G}$ that satisfies*

$$\rho(g \cdot x) = \rho(x) g^{-1} \qquad \forall x \in X, g \in \mathcal{G}.$$

Remark 2.3.1. *The definition of the moving frame map indicates that if $x_1, x_2 \in X$ and there exists $g \in \mathcal{G}$ such that $g \cdot x_1 = x_2$, then $\rho(x_1) \cdot x_1 = \rho(x_2) \cdot x_2$. Observe that*

$$\begin{aligned}
\rho(x_2) \cdot x_2 &= \rho(g \cdot x_1) \cdot (g \cdot x_1) \\
&= \rho(x_1) g^{-1} \cdot (g \cdot x_1) \\
&= \rho(x_1) \left(g^{-1} g\right) \cdot x_1 \\
&= \rho(x_1) \cdot x_1
\end{aligned}$$

Our focus is on moving frames for curves in $\mathbb{R}^2$ and curves and surfaces in $\mathbb{R}^3$ under the action of a group of geometric transformations. We make the following definitions for our purposes. The definitions are adopted from [2].

Definition 18 (Moving Frame Map for Curves)**.** *Let $\mathcal{G}$ be a group of geometric transformations on X, where $X = \mathbb{R}^2$ or $X = \mathbb{R}^3$. Let $\mathcal{J}^{(n)}(\mathcal{C}, X)$ be the space of n-jets of curves in X. A* moving frame map for curves *under the action of $\mathcal{G}$ on X is a map $\rho : \mathcal{J}^{(n)}(\mathcal{C}, X) \to \mathcal{G}$ that satisfies*

$$\rho\left(g \cdot \mathbf{z}^{(n)}\right) = \rho\left(\mathbf{z}^{(n)}\right) g^{-1} \qquad \forall g \in \mathcal{G}, \mathbf{z}^{(n)} \in \text{Domain}(\rho) \subset \mathcal{J}^{(n)}(\mathcal{C}, X),$$

for the prolonged $\mathcal{G}$-action on curves. Restricted to the n-jet of a curve $\mathcal{C}$, the map $\rho : \mathrm{j}^{(n)}(\mathcal{C}) \to \mathcal{G}$ is a moving frame map *for the curve $\mathcal{C}$.*

Definition 19 (Moving Frame Map for Surfaces)**.** *Let $\mathcal{G}$ be a group of geometric transformations on $\mathbb{R}^3$. Let $\mathcal{J}^{(n)}\left(\mathcal{S},\mathbb{R}^3\right)$ be the space of n-jets of surfaces in $\mathbb{R}^3$. A* moving frame map for surfaces *for the $\mathcal{G}$-action on $\mathbb{R}^3$ is a map $\rho : \mathcal{J}^{(n)}\left(\mathcal{S},\mathbb{R}^3\right) \to \mathcal{G}$ that satisfies*

$$\rho\left(g\cdot\mathbf{z}^{(n)}\right) = \rho\left(\mathbf{z}^{(n)}\right)g^{-1} \qquad \forall g\in\mathcal{G}, \mathbf{z}^{(n)} \in \mathrm{Domain}\left(\rho\right) \subset \mathcal{J}^{(n)}\left(\mathcal{S}, X\right)$$

for the prolonged $\mathcal{G}$-action on surfaces. Restricted to n-jet of a surface $\mathcal{S}$, the map $\rho : \mathrm{j}^{(n)}\left(\mathcal{S}\right) \to \mathcal{G}$ is a moving frame map *for the surface $\mathcal{S}$.*

Remark 2.3.2. *At times a moving frame map for curves or surfaces might only be defined for a subset of curves or surfaces, not for all curves or surfaces. A curve or surface that lies outside of the domain of a moving frame map can usually be characterized in another way.*

2.3.2 Cross-Sections, Normalization, Construction of Moving Frame Maps

For this section we will assume that $\mathcal{G}$ is a group of geometric transformations on $\mathbb{R}^2$ or $\mathbb{R}^3$. We will let X represent any of the following:

- The space of n-jets of curves in $\mathbb{R}^2$: $\mathcal{J}^{(n)}\left(\mathcal{C},\mathbb{R}^2\right)$
- The space of n-jets of curves in $\mathbb{R}^3$: $\mathcal{J}^{(n)}\left(\mathcal{C},\mathbb{R}^3\right)$
- The space of n-jets of surfaces in $\mathbb{R}^3$: $\mathcal{J}^{(n)}\left(\mathcal{S},\mathbb{R}^3\right)$

We will also use $\mathbf{z}^{(n)}$ to represent points in X. We typically define a moving frame ρ for the prolonged action of $\mathcal{G}$ to X by specifying a *cross-section*.

Definition 20 (Cross-section)**.** *A subspace $\mathcal{K}^{(n)} \subset X$ is said to be a* local cross-section *to the prolonged $\mathcal{G}$-action on X if for each $\mathbf{z}^{(n)} \in X$ the set of $g \in \mathcal{G}$ such that $g\cdot\mathbf{z} \in \mathcal{K}^{(n)}$ is discrete or empty.*

Remark 2.3.3. *We use a very weak definition of a cross-section here, but it is suitable for our purposes. Fels and Olver use a more technical definition for theoretical purposes. Their condition is meant to ensure that one has (suitable) theoretical control of solving the equation defined by $g\cdot\mathbf{z}^{(n)} \in \mathcal{K}^{(n)}$ for the group element $g \in \mathcal{G}$.*

Note that it is possible that there exist $\mathbf{z}^{(n)} \in X$ such that there does not exist any $g \in \mathcal{G}$ satisfying $g\cdot\mathbf{z}^{(n)} \in \mathcal{K}^{(n)}$. With a cross-section $\mathcal{K}^{(n)}$ for the prolonged action of $\mathcal{G}$ on X, we then implicitly define a moving frame map $\rho : X \to \mathcal{G}$ by

$$\rho\left(\mathbf{z}^{(n)}\right)\cdot\mathbf{z}^{(n)} \in \mathcal{K}^{(n)}. \tag{2.14}$$

In specific examples that we investigate, we are able to recover the moving frame map explicitly. Despite the possible issues in attempting to solve the equations defined by $\rho\left(\mathbf{z}^{(n)}\right)\cdot\mathbf{z}^{(n)} \in \mathcal{K}^{(n)}$, we take the point of view advocated for (under the appropriate hypotheses) by Fels and Olver in [2]:

Cross-sections $\leftrightarrow$ Moving frame maps ρ for the $\mathcal{G}$-action.

We will now outline the process for how to turn a (suitable) cross-section into a moving frame map. To do so, we will work abstractly in three-dimensions with curves. We will find all of our moving frame maps on $\mathcal{J}^{(1)}\left(\mathcal{C},\mathbb{R}^3\right)$, so we will illustrate the process in that setting.

Let $\mathcal{G}$ be a group of geometric transformations on $\mathbb{R}^3$ and prolong the action to $\mathcal{J}^{(1)}\left(\mathcal{C},\mathbb{R}^3\right)$. Assuming that we are working with curves $\mathcal{C}$ where $y = y(x)$ and $z = z(x)$ are functions of the independent variable x, then we have

$$\mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right) = \left\{\left(x,y,z,y',z'\right)\right\}.$$

Let the prolonged $\mathcal{G}$-action of $g \in \mathcal{G}$ on a (x,y,z,y',z') be given by

$$g\cdot\left(x,y,z,y',z'\right) = \left(\bar{x},\bar{y},\bar{z},\bar{y}',\bar{z}'\right)$$

and represented as

$$\begin{aligned}
\bar{x} &= \bar{x}\left(x,y,z,g\right)\\
\bar{y} &= \bar{y}\left(x,y,z,g\right)\\
\bar{z} &= \bar{z}\left(x,y,z,g\right)\\
\bar{y}' &= \bar{y}'\left(x,y,z,y',z',g\right)\\
\bar{z}' &= \bar{y}'\left(x,y,z,y',z',g\right).
\end{aligned}$$

Note that as before we are using $g \in \mathcal{G}$ to represent a group element that could depend on several parameters. We will define a cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$ by setting some set of the coordinates equal to constants, say

$$x = k_1, \qquad y = k_2, \qquad z = k_3, \qquad y' = k_4.$$

We then take a generic point $(x,y,z,y',z') \in \mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$ and attempt to find $g \in \mathcal{G}$ such that

$$g\cdot(x,y,z,y',z') = \left(\bar{x},\bar{y},\bar{z},\bar{y}',\bar{z}'\right) \in \mathcal{K}^{(1)}.$$

This gives a set of *normalization equations*

$$\bar{x} = \bar{x}\left(x,y,z,g\right) = k_1 \tag{2.15}$$
$$\bar{y} = \bar{y}\left(x,y,z,g\right) = k_2 \tag{2.16}$$
$$\bar{z} = \bar{z}\left(x,y,z,g\right) = k_3 \tag{2.17}$$
$$\bar{y}' = \bar{y}'\left(x,y,z,y',z',g\right) = k_4 \tag{2.18}$$

to be solved for the required element $g \in \mathcal{G}$. Provided that one can solve the normalization equations for g with some reasonable control over the solutions, the resulting map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right) \to \mathcal{G}$ defined by

$$\underbrace{\rho\left(x,y,z,y',z'\right)}_{\in\mathcal{G}}\cdot\underbrace{\left(x,y,z,y',z'\right)}_{\in\mathcal{J}^{(1)}(\mathbb{R},\mathbb{R}^2)} \in \mathcal{K}^{(1)}$$

will be a well-defined moving frame map.

Remark 2.3.4. *To be able to solve the normalization equations for $g \in \mathcal{G}$, one needs at least as many equations as parameters of the group $\mathcal{G}$.*

2.3.3 The Invariantization Process

We continue assuming $\mathcal{G}$ is a group of geometric transformations on $\mathbb{R}^2$ or $\mathbb{R}^3$ and that X can represent any of the following:

- The space of n-jets of curves in $\mathbb{R}^2$: $\mathcal{J}^{(n)}\left(\mathcal{C},\mathbb{R}^2\right)$
- The space of n-jets of curves in $\mathbb{R}^3$: $\mathcal{J}^{(n)}\left(\mathcal{C},\mathbb{R}^3\right)$
- The space of n-jets of surfaces in $\mathbb{R}^3$: $\mathcal{J}^{(n)}\left(\mathcal{S},\mathbb{R}^3\right)$

We now outline how a moving frame map $\rho : X \to \mathcal{G}$ for the prolonged $\mathcal{G}$-action on X can be used to construct invariants of curves/surfaces. The ability to easily construct invariants for curves and surfaces is the benefit of the Fels-Olver moving frame method.

Definition 21 (Differential Invariant)**.** *A* differential invariant *for the prolonged action of $\mathcal{G}$ on X is a function $F : X \to \mathbb{R}$ such that*

$$F\left(g \cdot \mathbf{z}^{(n)}\right) = F\left(\mathbf{z}^{(n)}\right) \qquad \forall g \in \mathcal{G}, \mathbf{z}^{(n)} \in X.$$

Restricted to the n-jet of a curve $\mathcal{C}$, the function $F\left(\mathrm{j}^{(n)}\left(\mathcal{C}\right)\right)$ is a differential invariant *of the curve $\mathcal{C}$.*

Definition 22 (Invariantization)**.** *Let $\rho : X \to \mathcal{G}$ be a moving frame map for the prolonged action of $\mathcal{G}$ on X and let $F : X \to \mathbb{R}$ be a differential function. The* invariantization of F *by the moving frame map ρ is the function $i\left(F\right) : X \to \mathbb{R}$ defined by*

$$i\left(F\right)\left(\mathbf{z}^{(n)}\right) := F\left(\rho\left(\mathbf{z}^{(n)}\right) \cdot \mathbf{z}^{(n)}\right).$$

The following theorem justifies the definition of invariantization.

Theorem 2.3.1. *Let $\rho : X \to \mathcal{G}$ be a moving frame map for the prolonged action of $\mathcal{G}$ on X and let $F : X \to \mathbb{R}$ be a differential function. The* invariantization of F *by the moving frame map ρ is a differential invariant for the prolonged action of $\mathcal{G}$ on X.*

Proof. The proof follows from the defining property of a moving frame and Remark 2.3.1. For $g \in \mathcal{G}$ and $\mathbf{z}^{(n)} \in X$ we have

$$\begin{aligned} i\left(F\right)\left(g \cdot \mathbf{z}^{(n)}\right) &:= F\left(\rho\left(g \cdot \mathbf{z}^{(n)}\right) \cdot \left(g \cdot \mathbf{z}^{(n)}\right)\right) \\ &= F\left(\rho\left(\mathbf{z}^{(n)}\right) g^{-1} \cdot \left(g \cdot \mathbf{z}^{(n)}\right)\right) \\ &= F\left(\rho\left(\mathbf{z}^{(n)}\right) \cdot \mathbf{z}^{(n)}\right) \\ &= i\left(F\right)\left(\mathbf{z}^{(n)}\right). \end{aligned}$$

□

Chapter 3

Curves in the Euclidean Plane

We begin by demonstrating how to use the Fels-Olver moving frame method by applying it to the Euclidean plane. Analysis of curves in the Euclidean plane using the Fels-Olver moving frame method has already been carried out in many different places and is the common example used to illustrate the method. See [2], [8], [11].

The point-set of the Euclidean plane is ordinary $\mathbb{R}^2$ and the geometry of the Euclidean plane is determined by the transformations generated by rotations and translations. We will consider curves $\mathcal{C}$ that can be described as the graph of a function $y = y(x)$. With sufficient assumptions on differentiability, this allows all curves except for vertical lines to be analyzed on open sets away from isolated singularities where $\frac{dy}{dx}$ is undefined on account of the tangent line being vertical. At such points, one could change their viewpoint and instead view the curve $\mathcal{C}$ locally as a graph where x is a function of y.

3.1 Geometric Transformations

The geometric transformations of the Euclidean plane are the familiar rotations and translations. We will represent the geometric transformations of Euclidean geometry as a matrix group and the action of the transformations will be given by matrix multiplication. To allow for this, we will identify points $p = (x, y)$ in the Euclidean plane with a point $\overrightarrow{p} = (x, y, 1)$ in three-dimensional space. Note that this identifies the Euclidean plane with the plane $z = 1$ in standard three-dimensional Cartesian space.

A Euclidean transformation of the plane is comprised of a rotation $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and a translation vector $\vec{v} = (a, b)$. Note that the rotation matrix R_θ is a rotation about the origin and the first column of the matrix is a point on the unit circle representing the rotation of the vector $\vec{e}_1 = (1, 0)$, while the second column in the matrix is a $\pi/2$ rotation of the vector $(\cos\theta, \sin\theta)$. The second column of the rotation matrix thus represents a corresponding rotation of the vector $\vec{e}_2 = (0, 1)$.

The group of Euclidean transformations on the plane can then be represented as a matrix Lie group as a

$$\mathcal{G}\left(\mathbb{E}^2\right) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \theta, a, b \in \mathbb{R} \right\}. \tag{3.1}$$

The transformation laws for the action of a matrix M in $\mathcal{G}\left(\mathbb{E}^2\right)$ on a point (x, y) are then given by matrix multiplication. Let $(\overline{x}, \overline{y}) = M \cdot (x, y)$ denote the transformation of a point (x, y) in the plane by a matrix M in $\mathcal{G}\left(\mathbb{E}^2\right)$. With the point (x, y) identified as $\vec{p} = (x, y, 1)$ and $\overrightarrow{p}^t$ denoting the transpose of $\vec{p}$, then the transformed point $(\overline{x}, \overline{y})$ is defined by

$$M\overrightarrow{p}^t = M \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \\ 1 \end{pmatrix}. \tag{3.2}$$

Substituting $M = \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix}$ into the above gives

$$\begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta + a \\ x\sin\theta + y\cos\theta + b \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \\ 1 \end{pmatrix}. \tag{3.3}$$

In total, for a given geometric transformation M using the parameters (a, b, θ), the coordinates of the transformed point $(\overline{x}, \overline{y}) = M \cdot (x, y)$ are

$$\bar{x} = x\cos\theta - y\sin\theta + a \tag{3.4}$$

$$\bar{y} = x\sin\theta + y\cos\theta + b. \tag{3.5}$$

Determining the effect of a transformation M on a curve $\mathcal{C}$ that is described by the graph of a function $y = y(x)$ follows by restricting the points (x, y) in the plane to points of the form $(x, y(x))$. That is, if $\mathcal{C} = \left\{(x, y(x)) \middle| x \in \mathbb{R}\right\}$, then $M \cdot \mathcal{C}$ consists of all points of the form

$$(\overline{x}, \overline{y}) = \left(x\cos\theta - y(x)\sin\theta + a, x\sin\theta + y(x)\cos\theta + b\right). \tag{3.6}$$

Note that if $\mathcal{C}$ is the graph of a function $y = y(x)$, then the curve $M \cdot \mathcal{C}$ need not be the graph of a function of x as a transformation could rotate the curve $\mathcal{C}$ so that it can no longer be described as a graph of the independent variable x.

3.2 Prolonged Transformations

Viewing the curve $\mathcal{C}$ described by the function $y = y(x)$ as a geometric object, the transformations in $\mathcal{G}\left(\mathbb{E}^2\right)$ induce prolonged transformations on the derivatives of the function y. Geometrically this is equivalent to transformations of the infinitesimal approximations from the Taylor series expansion. For example, a transformation M in $\mathcal{G}\left(\mathbb{E}^2\right)$ will take the tangent line of $\mathcal{C}$ at the point $(x, y(x))$ to the tangent line of $M \cdot \mathcal{C}$ at the point $(\overline{x}, \overline{y(x)})$, the approximating quadratic function of $\mathcal{C}$ at the point $(x, y(x))$ to the approximating quadratic of the transformed curve of $M \cdot \mathcal{C}$ at the point $(\overline{x}, \overline{y(x)})$, and so on and so forth for higher order approximations.

Following Definition 13 the prolonged transformations on derivatives can be found by applying the prolonged transformation operator to the above transformation laws. Applying the implicit differentiation operator (2.4) to (3.4) gives

$$D_x\bar{x} = D_x\left(x\cos\theta - y\sin\theta + a\right) = \cos\theta - y'\sin\theta.$$

The prolonged transformation operator for the group action is then

$$\mathcal{D}_x = \frac{D_x}{D_x\bar{x}} = \frac{1}{\cos\theta - y'\sin\theta}D_x.$$

Applying the prolonged transformation operator $\mathcal{D}_x$ to the transformed coordinates of a curve $\mathcal{C}$ described by the graph of a function as in (3.6) we find the transformed first derivative to be

$$\bar{y}' = \mathcal{D}_x\bar{y} = \frac{\sin\theta + y'\cos\theta}{\cos\theta - y'\sin\theta}.$$

Note that applied to $\bar{y}$, $\mathcal{D}_x\bar{y} = \frac{D_x\bar{y}}{D_x\bar{x}} = \frac{d\bar{y}}{d\bar{x}}$ gives the derivative of the transformed curve where the transformed variable $\overline{y}$ is viewed as a function of the transformed variable $\overline{x}$.

The prolonged transformation of derivatives of higher order are defined inductively. Namely,

$$\bar{y}^{(n)} = \mathcal{D}_x(\bar{y}^{(n-1)}), \tag{3.7}$$

where $\bar{y}^{(n-1)}$ is the transformed $(n-1)^{th}$ derivative. Up to the third order, we have the prolonged derivatives of curves in the Euclidean plane listed out below:

$$\bar{y}' = \mathcal{D}_x\bar{y} = \frac{\sin\theta + y'\cos\theta}{\cos\theta - y'\sin\theta} \tag{3.8}$$

$$\bar{y}'' = \mathcal{D}_x\bar{y}' = \frac{y''}{(\cos\theta - y'\sin\theta)^3} \tag{3.9}$$

$$\bar{y}^{(3)} = \mathcal{D}_x\bar{y}'' = \frac{3\sin\theta\,(y'')^2 + (\cos\theta - y'\sin\theta)y^{(3)}}{(\cos\theta - y'\sin\theta)^5} \tag{3.10}$$

Continuing to apply the prolonged transformation operator $\mathcal{D}_x$ to higher order derivatives gives the prolonged transformation rules for the action of $\mathcal{G}\left(\mathbb{E}^2\right)$ on $\mathcal{J}^{(n)}(\mathbb{R},\mathbb{R})$, the space of n-jets of order n.

3.3 Normalization and The Moving Frame

With prolonged transformation rules for $\mathcal{G}\left(\mathbb{E}^2\right)$ on $\mathcal{J}^{(n)}(\mathbb{R},\mathbb{R})$ at hand, a moving frame map can then be constructed by using a cross section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}(\mathbb{R},\mathbb{R})$. We elect to use the cross-section that corresponds to the traditional moving frame method for curves in the Euclidean plane and will produce the traditional representation of invariants of curves. See [13] for comparison.

Let $\mathcal{C}$ be a curve described the graph of a function $(x, y(x))$ with one-jet $(x, y(x), y'(x)) = (x, y, y')$. At an arbitrary point point (x, y, y') along the curve $\mathcal{C}$, we will put the curve in normalized position by solving for the transformation M in $\mathcal{G}\left(\mathbb{E}^2\right)$ that will bring the point (x, y, y') to the point

$(0,0,0)$. That is, we look for the transformation M in $\mathcal{G}\left(\mathbb{E}^2\right)$ such that $M\cdot(x,y,y')=(\overline{x},\overline{y},\overline{y}')=(0,0,0)$. Combined with (3.6) and (3.8), this results in the normalization equations.

$$\overline{x} = x\cos\theta - y(x)\sin\theta + a = 0 \tag{3.11}$$

$$\overline{y} = x\sin\theta + y(x)\cos\theta + b = 0 \tag{3.12}$$

$$\overline{y}' = \frac{\sin\theta + y'\cos\theta}{\cos\theta - y'\sin\theta} = 0. \tag{3.13}$$

The corresponding cross section is thus $\mathcal{K}^{(1)} = \left\{(x,y,y')\in\mathcal{J}^{(1)}\middle| x=y=y'=0\right\}$. Note that the normalization equations amount to bringing a point (x,y) on a curve $\mathcal{C}$ to the origin and then rotating the curve until the tangent line is horizontal.

Solving (3.11), (3.12), (3.13) for the variables a,b, and θ yields

$$a = -(x\cos\theta - y\sin\theta) \tag{3.14}$$

$$b = -(x\sin\theta + y\cos\theta) \tag{3.15}$$

$$\theta = \tan^{-1}\left(-y'\right) \tag{3.16}$$

Note that with $\theta = \tan^{-1}(-y')$, we can use properties of trig functions to find simplified expressions of $\cos\theta$ and $\sin\theta$. Specifically, we have

$$\cos\theta = \frac{1}{\sqrt{1+(y')^2}} \qquad \text{and} \qquad \sin\theta = -\frac{y'}{\sqrt{1+(y')^2}}. \tag{3.17}$$

It then follows that

$$a = -\frac{x+yy'}{\sqrt{1+(y')^2}} \tag{3.18}$$

$$b = -\frac{xy'-y}{\sqrt{1+(y')^2}} \tag{3.19}$$

$$\theta = \tan^{-1}\left(-y'\right) \tag{3.20}$$

Having the transformation parameters a,b, and θ defined in terms of the components of an arbitrary point (x,y,y') of the one-jet of $\mathcal{C}$ defines the moving frame map $\rho : \mathrm{j}^{(1)}\left(\mathcal{C}\right)\to\mathcal{G}\left(\mathbb{E}^2\right)$, with

$$\rho(x,y,y') = \begin{pmatrix} \frac{1}{\sqrt{1+(y')^2}} & \frac{y'}{\sqrt{1+(y')^2}} & -\frac{x+yy'}{\sqrt{1+(y')^2}} \\ -\frac{y'}{\sqrt{1+(y')^2}} & \frac{1}{\sqrt{1+(y')^2}} & \frac{xy'-y}{\sqrt{1+(y')^2}} \\ 0 & 0 & 1 \end{pmatrix} = M \in \mathcal{G}\left(\mathbb{E}^2\right). \tag{3.21}$$

3.4 Invariantization

With solutions to the normalization equations (3.11), (3.12), (3.13) and the moving frame map (3.21) we can now substitute the expressions for a,b, and θ in terms of x,y, and y' into the prolonged transformations of the higher order derivatives to obtain differential invariants of any desired order.

We will denote the invariants obtained by making the indicated substitutions into (3.7) by $i(\overline{y}^{(n)})$. Below, we indicate the differential invariants of order two and three obtained by making the indicated substitutions into (3.9) and (3.10). It is more convenient to substitute the expressions for $\cos\theta$ and $\sin\theta$ from (3.17), than to substitute the expression for θ.

$$
\begin{aligned}
i(\overline{y}'') &= \frac{y''}{(1+(y')^2)^{\frac{3}{2}}}\\
i(\overline{y}^{(3)}) &= \frac{-3y'\,(y'')^2 + y^{(3)} + (y')^2\,y^{(3)}}{(1+(y')^2)^3}
\end{aligned}
$$

Remark 3.4.1. *Note that the invariant* $i(\overline{y}'') = \dfrac{y''}{(1+(y')^2)^{\frac{3}{2}}}$ *gives the familiar expression for the curvature of a curve in the plane that is (can be) calculated in a standard calculus course.*

Remark 3.4.2. *Note that as one might expect based on geometric intuition and familiarity with calculus, the transformation parameters corresponding to translation in the Euclidean plane (i.e., a and b) do not show up in the prolonged transformations of the derivatives of curves.*

Chapter 4

Curves in the Lorentz-Minkowski Plane

We now turn our attention to the geometry of the Lorentz-Minkowski plane and apply the Fels-Olver moving frame method to catalog the differential invariants of curves in the indicated geometry. Traditional moving frame methods have been carried out in the Lorentz-Minkowski plane and Lorentz-Minkowski three-space. They often make homework problems in advanced differential geometry classes. See [6] and [7]. To the best of our knowledge, the application of the Fels-Olver moving frame method to identify differential invariants of curves in the Lorentz-Minkowski plane is new.

The point-set for the Lorentz-Minkowski plane is the two-dimensional Cartesian plane. The feature that distinguishes the Lorentz-Minkowski plane from the Euclidean plane is the collection of geometric transformations. As before, we consider curves $\mathcal{C}$ that can be described as the graph of a function $y = y(x)$. As with potential isolated singularities when the derivative $y' = \frac{dy}{dx}$, our analysis will also reveal a singularity in the construction of the moving frame map when $\left|\dfrac{dy}{dx}\right| = 1$. Curves in the Lorentz-Minkowski plane where $\frac{dy}{dx}$ is constant of ± 1 are well-understood: they correspond to null lines in the geometry of general relativity and separate spacelike curves from timelike curves [12].

4.1 Geometric Transformations

We will denote the Lorentz-Minkowski plane by $\mathbb{L}^2$ and we will follow closely with the identifications used in the analysis of curves in the Euclidean plane. The distinguishing feature between $\mathbb{L}^2$ and $\mathbb{E}^2$ are the geometric transformations. The geometric transformations defining $\mathbb{L}^2$ consist of three parameters a, b, and θ. The action of the transformations on the plane can be realized by matrix multiplication.

As before we will identify individual points $p = (x, y)$ in the plane with the point $\overrightarrow{p} = (x, y, 1)$ in three-dimensional space. This identifies the Lorentz-Minkowski plane with the plane $z = 1$ in standard three-dimensional Cartesian space.

A geometric transformation of the Lorentz-Minkowski plane is comprised of a hyperbolic rotation $R_\theta = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$ and a translation vector $\vec{v} = (a, b)$. Note that the hyperbolic rotation matrix R_θ is a hyperbolic rotation about the origin. The first column of the matrix is a point on the hyperbola $x^2 - y^2 = 1$ with $x > 0$ and represents the image of the vector $\vec{e}_1 = (1, 0)$ under the indicated rotation, while the second column in the matrix is the image of the vector $\vec{e}_2 = (0,1)$ under the same rotation. Note that the image of $\vec{e}_2 = (0,1)$ under the hyperbolic rotation will like on the hyperbola $y^2 - x^2 = 1$.

The group of geometric transformations on the Lorentz-Minkowski plane[1] can then be represented as a matrix group by

$$\mathcal{G}\left(\mathbb{L}^2\right) = \left\{ \begin{pmatrix} \cosh\theta & \sinh\theta & a \\ \sinh\theta & \cosh\theta & b \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, \theta \in \mathbb{R} \right\}. \tag{4.1}$$

The transformation laws for the action of a matrix M in $\mathcal{G}\left(\mathbb{L}^2\right)$ on a point (x, y) are then given by matrix multiplication. Let $M \cdot (x, y) = (\overline{x}, \overline{y})$ denote the transformation of a point (x, y) in the plane by a matrix M in $\mathcal{G}\left(\mathbb{L}^2\right)$. With the point (x, y) identified as $\vec{p} = (x, y, 1)$ and $\overrightarrow{p}^t$ denoting the transpose of $\vec{p}$, then the transformed point $(\overline{x}, \overline{y})$ is defined by

$$M\overrightarrow{p}^t = M \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \\ 1 \end{pmatrix}. \tag{4.2}$$

Substituting $M = \begin{pmatrix} \cosh\theta & \sinh\theta & a \\ \sinh\theta & \cosh\theta & b \\ 0 & 0 & 1 \end{pmatrix}$ into the above gives

$$\begin{pmatrix} \cosh\theta & \sinh\theta & a \\ \sinh\theta & \cosh\theta & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x\cosh\theta + y\sinh\theta + a \\ x\sinh\theta + y\cosh\theta + b \\ 1 \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \\ 1 \end{pmatrix}. \tag{4.3}$$

This results in the following transformation laws for a point (x, y) transformed to a point $(\overline{x}, \overline{y})$ by the parameters (a, b, θ):

$$\bar{x} = x\cosh\theta + y\sinh\theta + a \tag{4.4}$$

$$\bar{y} = x\sinh\theta + y\cosh\theta + b. \tag{4.5}$$

In an identical manner to the analysis of curves in the Euclidean plane, we study the effect of a transformation $M \in \mathcal{G}\left(\mathbb{L}^2\right)$ on a curve $\mathcal{C}$ given by the graph of a function $y = y(x)$ by substituting $y = y(x)$ into (4.4) and (4.5).

4.2 Prolonged Transformations

Following Section 2.2.1, we now calculate the prolongation of the transformations in $\mathcal{G}\left(\mathbb{L}^2\right)$ to the derivatives of the function $y = y(x)$ describing a curve $\mathcal{C}$. In an identical manner to the analysis carried out on the Euclidean plane, this is geometrically equivalent to the transformations of the infinitesimal approximations from the Taylor series expansion. The main difference is that we are

[1] Similar to Euclidean geometry, there is the option to include reflections in the collection of geometric transformations. For simplicity of exposition, we elect to work with hyperbolic rotations and translations. This ignores potential issues with orientations, but such issues can be resolved.

not as accustomed to the transformations in $\mathcal{G}\left(\mathbb{L}^2\right)$ as we are accustomed to the traditional rotations and translations of Euclidean geometry.

Applying the implicit differentiation operator (2.4) to (4.4) we have that the prolonged transformation operator $\mathcal{D}_x$ is given by

$$\mathcal{D}_x = \frac{D_x}{D_x \bar{x}} = \frac{1}{\cosh\theta + y' \sinh\theta} \mathcal{D}_x, \tag{4.6}$$

where

$$D_x = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + y'' \frac{\partial}{\partial y'} + \dots$$

is the implicit differentiation operator and

$$D_x \bar{x} = \mathcal{D}_x \left(x \cosh\theta + y \sinh\theta + a\right) = \cosh\theta + y' \sinh\theta.$$

Remark 4.2.1. *In the same manner as Euclidean geometry, $\mathcal{D}_x \bar{y} = \frac{D_x \bar{y}}{D_x \bar{x}} = \frac{d\bar{y}}{d\bar{x}}$ gives the derivative of the transformed curve where the transformed variable $\overline{y}$ is viewed as a transformation of the transformed variable $\overline{x}$. The connection to the geometry in consideration stems from the fact that $\mathcal{D}_x$ depends on the transformations defining the geometry.*

Applying the prolonged transformation operator (4.6) to $\bar{y}$ from (4.5) we obtain

$$\bar{y}' = \frac{\sinh\theta + y' \cosh\theta}{\cosh\theta + y' \sinh\theta}. \tag{4.7}$$

The transformation of any higher order derivative can be calculated by repeated application of $\mathcal{D}_x$ (4.6). The transformations of the second and third order derivatives are recorded below.

$$\bar{y}'' = \frac{y''}{\left(\cosh\theta + y' \sinh\theta\right)^3} \tag{4.8}$$

$$\bar{y}^{(3)} = \frac{-3y''^2 \sinh\theta + (\cosh\theta + y' \sinh\theta) y^{(3)}}{\left(\cosh\theta + y' \sinh\theta\right)^5} \tag{4.9}$$

The fundamental differential invariant of curves in $\mathbb{L}^2$ will come from invariantizing (4.8). This invariant will correspond to the curvature of a curve in the Lorentz-Minkowski plane.

The construction of the moving frame map is split up into two separate cases depending on whether $|y'| < 1$ or $|y'| > 1$.

Remark 4.2.2. *In the following sections, we will be covering how the invariants change depending on the value of $|y'|$. There are null lines in the Lorentz-Minkowski plane at $y = \pm x + c$, where c is any constant. Singularities with the moving frame along a curve $\mathcal{C}$ will develop where $y' = \pm 1$. Thus, for curves where $y' = \pm 1$, their slope is considered an invariant condition, in that performing any geometric transformations on these curves will not change the value of $\overline{y}'$. We will develop two moving frames as a result. One moving frame when $|\overline{y}'| < 1$ and another when $|\overline{y}'| > 1$. Curves are then analyzed on open intervals depending on the magnitude of $|\overline{y}|$.*

4.3 Normalization and The Moving Frame

4.3.1 $|y'| < 1$

In this section we assume that $\mathcal{C}$ is described by the graph of a function $y = y(x)$ that satisfies $|y'(x)| < 1$ for all x in the domain. We construct our moving frame map by using the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}(\mathbb{R}, \mathbb{R})$ defined by the equations

$$x = 0, \qquad y = 0, \qquad \textit{and} \qquad y' = 0.$$

Working along a curve $\mathcal{C}$ described by the graph $(x, y(x))$ with one-jet $(x, y(x), y'(x)) = (x, y, y')$ we aim to determine the transformation M (depending on x) in $\mathcal{G}\left(\mathbb{L}^2\right)$ that will bring the one-jet to the point $(0, 0, 0)$. That is, we look for the transformation M in $\mathcal{G}\left(\mathbb{L}^2\right)$ such that $M \cdot (x, y, y') = (\overline{x}, \overline{y}, \overline{y}') = (0, 0, 0)$. See Figure 4.1. The corresponding normalization equations are

$$\bar{x} = x \cosh\theta + y \sinh\theta + a = 0 \tag{4.10}$$

$$\bar{y} = x \sinh\theta + y \cosh\theta + b = 0 \tag{4.11}$$

$$\bar{y}' = \frac{\sinh\theta + y' \cosh\theta}{\cosh\theta + y' \sinh\theta} = 0. \tag{4.12}$$

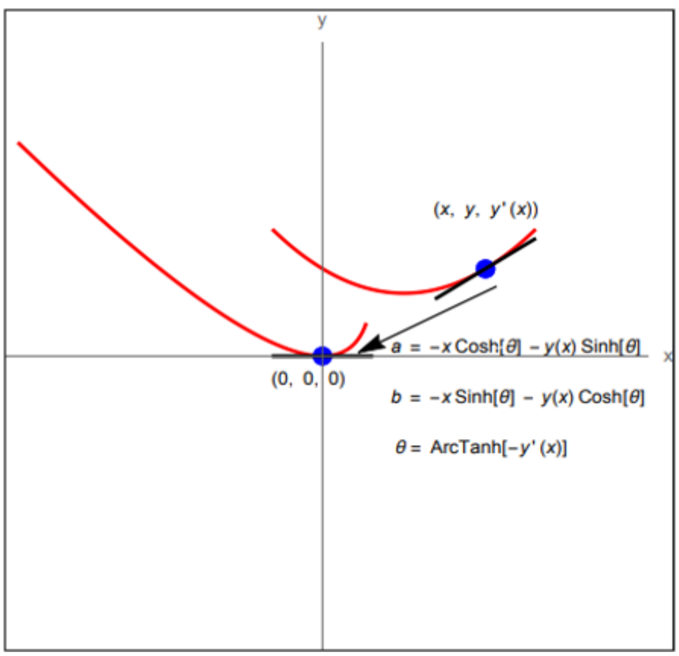

Figure 4.1: The normalization of a point $(x, y, y'(x))$ on a curve with $|y'(x)| < 1$ in the construction of the moving frame map. The curve is translated to the origin and then a hyperbolic rotation is applied to make the tangent line horizontal.

Solving for the variables a, b, and θ in a similar fashion as with the Euclidean curves, we find that

$$\begin{aligned} a &= -\,x\cosh\theta - y\sinh\theta && (4.13)\\ b &= -\,x\sinh\theta - y\cosh\theta && (4.14)\\ \theta &= \tanh^{-1}(-y') && (4.15)\end{aligned}$$

Remark 4.3.1. *Solving for θ stems from the observation that if $|y'| < 1$, then $|\bar{y}'| < 1$. We can set the prolonged transformation of the derivative $\bar{y}' = \frac{\sinh\theta + y'\cosh\theta}{\cosh\theta + y'\sinh\theta}$ equal to some constant λ, then solve for θ. The equation then becomes*

$$\sinh\theta + y'\cosh\theta - \lambda\cosh\theta - y'\lambda\sinh\theta = 0.$$

Using the definitions of $\sinh\theta$ and $\cosh\theta$ in terms of exponential function, this results in

$$\begin{aligned} (1 - y'\lambda)(e^{\theta} - e^{-\theta}) - (y' - \lambda)(e^{\theta} + e^{-\theta}) = 0\\ e^{2\theta}(1 + y' - \lambda - y'\lambda) - ((1 - y')(1 + \lambda)) = 0\\ e^{2\theta} = \frac{(1 - y')(1 + \lambda)}{(1 + y')(1 - \lambda)}\\ \theta = \frac{1}{2}\ln\left(\frac{(1 - y')(1 + \lambda)}{(1 + y')(1 - \lambda)}\right). \end{aligned}$$

This equation for θ is now dependent on whatever constant λ is chosen when resolving $\bar{y} = \lambda$. In order for this equation to hold, the argument of the natural log must be positive. As y' is assumed to be $-1 < y' < 1$, both factors contained in the argument are positive numbers. The only way for this to remain true is if λ remains in the same bounds as y'. As a result, when $|y'| < 1$, we can only use a λ such that $|\lambda| < 1$ as well. For similar reasons, if $|y'| > 1$, then $|\lambda| > 1$ as well.

As in the Euclidean case, standard properties of hyperbolic trig functions allow us to find reduced expressions for $\cosh\theta$ and $\sinh\theta$ when $\theta = \tanh^{-1}(-y')$. By substituting the value of θ into the hyperbolic trig functions, we find that

$$\begin{aligned} \cosh\theta &= \frac{1}{\sqrt{1 - (y')^2}}\\ \sinh\theta &= -\frac{y'}{\sqrt{1 - (y')^2}}. \end{aligned}$$

The parameters a and b are subsequently reduced down to

$$a = \frac{yy' - x}{\sqrt{1 - (y')^2}} \qquad \textit{and} \qquad b = \frac{xy' - y}{\sqrt{1 - (y')^2}}.$$

It follows that solving the normalization equations for the parameters a, b, and θ yields

$$\begin{aligned} a &= \frac{yy' - x}{\sqrt{1 - (y')^2}}\\ b &= \frac{xy' - y}{\sqrt{1 - (y')^2}}\\ \theta &= \tanh^{-1}\left(-y'\right). \end{aligned}$$

Having the transformation parameters a, b, and θ defined in terms of the components of an arbitrary point (x, y, y') of the one-jet of $\mathcal{C}$ defines the moving frame map $\rho : \mathrm{j}^{(1)}(\mathcal{C}) \to \mathcal{G}(\mathbb{L}^2)$, with

$$\rho(x, y, y') = \begin{pmatrix} \frac{1}{\sqrt{1-(y')^2}} & \frac{-y'}{\sqrt{1-(y')^2}} & \frac{yy'-x}{\sqrt{1-(y')^2}} \\ \frac{-y'}{\sqrt{1-(y')^2}} & \frac{1}{\sqrt{1-(y')^2}} & \frac{xy'-y}{\sqrt{1-(y')^2}} \\ 0 & 0 & 1 \end{pmatrix} = M \in \mathcal{G}(\mathbb{L}^2)\,. \tag{4.16}$$

4.3.2 $|y'| > 1$

We now consider the case where the function $y = y(x)$ defining the curve satisfies $|y'(x)| > 1$ for x in the domain. We construct our moving frame map by using the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}(\mathbb{R}, \mathbb{R})$ determined by

$$x = 0, \qquad y = 0, \qquad \textit{and} \qquad y' = 2.$$

We look for a transformation M (depending on x) in $\mathcal{G}(\mathbb{L}^2)$ that will bring the one-jet to the point $(0, 0, 2)$. That is, we want to find a transformation M in $\mathcal{G}(\mathbb{L}^2)$ such that $M \cdot (x, y, y') = (\overline{x}, \overline{y}, \overline{y}') = (0, 0, 2)$. See Figure 4.2.

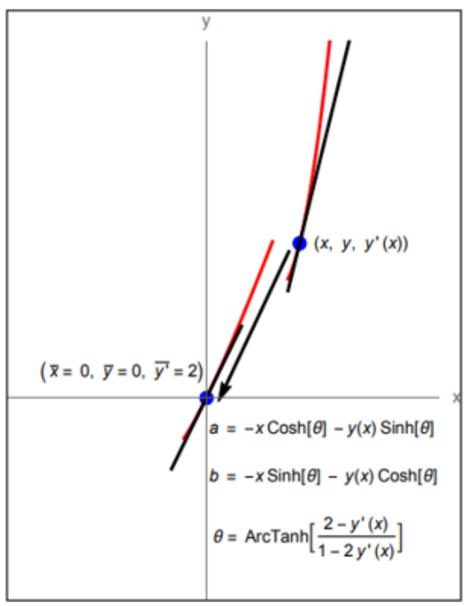

Figure 4.2: The normalization of a point $(x, y, y'(x))$ on a curve with $|y'(x)| > 1$ in the construction of the moving frame map. The curve is translated to the origin and then a hyperbolic rotation is applied so that the tangent line has a slope of 2.

The corresponding normalization equations are

$$\bar{x} = x \cosh\theta + y \sinh\theta + a = 0 \tag{4.17}$$

$$\bar{y} = x \sinh\theta + y \cosh\theta + b = 0 \tag{4.18}$$

$$\bar{y}' = \frac{\sinh\theta + y' \cosh\theta}{\cosh\theta + y' \sinh\theta} = 2. \tag{4.19}$$

Solving for the variables a, b, and θ in a similar fashion as with the Euclidean curves, we find that

$$a = -\, x \cosh\theta - y \sinh\theta \tag{4.20}$$
$$b = -\, x \sinh\theta - y \cosh\theta \tag{4.21}$$
$$\theta = \tanh^{-1}\left(\frac{2-y'}{1-2y'}\right). \tag{4.22}$$

Again, properties of hyperbolic trig functions allow us to find reduced expressions for $\cosh\theta$ and $\sinh\theta$ when $\theta = \tanh^{-1}\left(\frac{2-y'}{1-2y'}\right)$. By substituting the value of θ into the hyperbolic trig functions, we find that

$$\cosh\theta = \frac{-1+2y'}{\sqrt{3}\sqrt{-1+(y')^2}}$$
$$\sinh\theta = \frac{-2+y'}{\sqrt{3}\sqrt{-1+(y')^2}}$$

The parameters a and b are subsequently reduced to

$$a = \frac{x+2y-(2x+y)y'}{\sqrt{3}\sqrt{-1+(y')^2}} \qquad \textit{and} \qquad b = \frac{2x+y-(x+2y)y'}{\sqrt{3}\sqrt{-1+(y')^2}}.$$

It follows that solving the normalization equations for the parameters a, b, and θ yields

$$a = \frac{x+2y-(2x+y)y'}{\sqrt{3}\sqrt{-1+(y')^2}}$$
$$b = \frac{2x+y-(x+2y)y'}{\sqrt{3}\sqrt{-1+(y')^2}}$$
$$\theta = \tanh^{-1}\left(\frac{2-y'}{1-2y'}\right).$$

Having the transformation parameters a, b, and θ defined in terms of the components of an arbitrary point (x, y, y') of the one-jet of $\mathcal{C}$ defines the moving frame map $\rho : \mathrm{j}^{(1)}(\mathcal{C}) \to \mathcal{G}\left(\mathbb{L}^2\right)$, with

$$\rho(x,y,y') = \begin{pmatrix} \frac{-1+2y'}{\sqrt{3}\sqrt{-1+(y')^2}} & \frac{-2+y'}{\sqrt{3}\sqrt{-1+(y')^2}} & \frac{x+2y-(2x+y)y'}{\sqrt{3}\sqrt{-1+(y')^2}} \\ \frac{-2+y'}{\sqrt{3}\sqrt{-1+(y')^2}} & \frac{-1+2y'}{\sqrt{3}\sqrt{-1+(y')^2}} & \frac{2x+y-(x+2y)y'}{\sqrt{3}\sqrt{-1+(y')^2}} \\ 0 & 0 & 1 \end{pmatrix} = A \in \mathcal{G}\left(\mathbb{L}^2\right). \tag{4.23}$$

4.4 Invariantization

We now produce the fundamental differential invariants for curves in the Lorentz-Minkowski plane.

4.4.1 $|y'| < 1$

Substituting values of $a, b, \cosh\theta$, and $\sinh\theta$ from the moving frame map in (4.16) back into the prolonged transformations of the derivatives (4.8) and (4.9), we obtain the invariants:

$$i(\bar{y}'') = \frac{y''}{(1-(y')^2)^{\frac{3}{2}}} \tag{4.24}$$

$$i(\bar{y}^{(3)}) = \frac{-3y'(y'')^2 - y^{(3)} + (y')^2 y^{(3)}}{(1-(y')^2)^3}. \tag{4.25}$$

All higher order differential invariants can be obtained via a combination of applying (4.6) to the transformed derivatives and making the substitutions from the components of the moving frame map (4.16).

Remark 4.4.1. *Note the similarity between the invariant (4.24) for curves in the Lorentz-Minkowski plane and the curvature of curves in the Euclidean plane.*

4.4.2 $|y'| > 1$

Substituting values of $a, b, \cosh\theta$, and $\sinh\theta$ from the moving frame map in (4.23) back into the prolonged transformations of the derivatives (4.8) and (4.9), we obtain the following invariants for curves $y = y(x)$ with $|y'| > 1$:

$$i(\bar{y}'') = \frac{3\sqrt{3}y''}{(-1+(y')^2)^{\frac{3}{2}}} \tag{4.26}$$

$$i(\bar{y}^{(3)}) = \frac{-27(-2+y')(y'')^2 + 9y^{(3)}((y')^2-1)}{(1-(y')^2)^3}. \tag{4.27}$$

Remark 4.4.2. *The second order invariant (4.26) can be multiplied by $\frac{1}{3\sqrt{3}}$ to be consistent with (4.24). Any constant multiple (or invariant multiple) of a differential invariant will be an invariant.*

Chapter 5

Curves and Surfaces in $\mathcal{H}^3{}_R$

In this chapter we find differential invariants of curves and surfaces in the three-dimensional Heisenberg group $\mathcal{H}_R^3$. The point set is the ordinary three-dimensional Cartesian space $\mathbb{R}^3$. The geometry is defined by the geometric transformations described below. There are similarities between the geometric transformations defining the geometry of $\mathcal{H}_R^3$ and the transformations of the Euclidean plane $\mathbb{E}^2$. These similarities will result in invariants of curves in the Euclidean plane being invariants of curves in $\mathcal{H}_R^3$ as well.

5.1 Geometric Transformations

The group of geometric transformations defining the geometry of $\mathcal{H}_R^3$ is the subgroup of $GL_4(\mathbb{R})$ defined by

$$\mathcal{G}\left(\mathcal{H}_R^3\right) = \left\{ \left. \begin{pmatrix} \cos\theta & -\sin\theta & 0 & a \\ \sin\theta & \cos\theta & 0 & b \\ \frac{1}{2}\left(a\sin\theta - b\cos\theta\right) & \frac{1}{2}\left(a\cos\theta + b\sin\theta\right) & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| a, b, c, \theta \in \mathbb{R} \right\}. \tag{5.1}$$

Identifying $\mathbf{z} = (x, y, z) \in \mathcal{H}_R^3$ with $\mathbf{z} = (x, y, z, 1)^t \in \mathbb{R}^4$, then the action of $A \in \mathcal{G}\left(\mathcal{H}_R^3\right)$ on $\mathcal{H}_R^3 \sim \mathbb{R}^3$ is given by $A\mathbf{z} = \overline{\mathbf{z}} = (\overline{x}, \overline{y}, \overline{z}, 1)^t$, where

$$\overline{x} = x\cos\theta - y\sin\theta + a \tag{5.2}$$

$$\overline{y} = x\sin\theta + y\cos\theta + b \tag{5.3}$$

$$\overline{z} = z + c + \frac{1}{2}\left(a\sin\theta - b\cos\theta\right)x + \frac{1}{2}\left(a\cos\theta + b\sin\theta\right)y. \tag{5.4}$$

5.2 Curves

We look at curves $\mathcal{C}$ in $\mathbb{R}^3$ where $y = y(x)$ and $z = z(x)$ are described as functions of the independent variable x. The space of n-jets are described by

$$\mathcal{J}^{(n)}\left(\mathbb{R}, \mathbb{R}^2\right) = \left\{ \left(x, y, z, y', z', y'', z'', \ldots, y^{(n)}, z^{(n)}\right) \right\} \sim \mathbb{R}^{3+2n}.$$

5.2.1 Prolonged Transformations

We we will represent a point in $\mathcal{J}^{(n)}\left(\mathbb{R},\mathbb{R}^2\right)$ by $\mathbf{z}^{(n)} = \left(x, y, z, y', z', y'', z'', \ldots, y^{(n)}, z^{(n)}\right)$, where coordinates are named so that the indicated derivatives of the dependent variables y and z are taken with respect to the independent variable x. The implicit differential operator D_x is given by Definition 12. We find the prolonged transformation operator from Definition 13 by first applying D_x to (5.2), giving

$$D_x\overline{x} = D_x\left(x\cos\theta - y\sin\theta + a\right) = \cos\theta - y'\sin\theta.$$

The prolonged transformation operator is

$$\mathcal{D}_x = \frac{1}{\left(\cos\theta - y'\sin\theta\right)} D_x. \tag{5.5}$$

Applying the prolonged transformation operator (5.5) to (5.3) and (5.4) gives the prolonged action of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ to $\mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$. We have

$$\overline{y}' = \mathcal{D}_x\overline{y} = \frac{\sin\theta + y'\cos\theta}{\cos\theta - y'\sin\theta} \tag{5.6}$$

$$\overline{z}' = \mathcal{D}_x\overline{z} = \frac{2z' + \left(a\sin\theta - b\cos\theta\right) + \left(a\cos\theta + b\sin\theta\right)y'}{2\left(\cos\theta - y'\sin\theta\right)}. \tag{5.7}$$

The prolongation of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ to higher order jet spaces can be obtained by repeated application of $\mathcal{D}_x$:

$$\overline{y}^{(k)} = \mathcal{D}_x^k\left(\overline{y}\right) = \mathcal{D}_x\left(\overline{y}^{(k-1)}\right) \qquad \text{and} \qquad \overline{z}^{(k)} = \mathcal{D}_x^k\left(\overline{z}\right) = \mathcal{D}_x\left(\overline{z}^{(k-1)}\right).$$

To order two we have

$$\overline{y}'' = \frac{y''}{\left(\cos\theta - y'\sin\theta\right)^3} \qquad \textit{and} \qquad \overline{z}'' = \frac{y''\left(a + 2z'\sin\theta\right) + 2z''\left(\cos\theta - y'\sin\theta\right)}{2\left(\cos\theta - y'\sin\theta\right)^3}. \tag{5.8}$$

5.2.2 Normalization and The Moving Frame

We now obtain a moving frame for the action of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ on curves by using the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$ determined by setting $x = 0$, $y = 0$, $z = 0$ and $y' = 0$. The resulting moving frame map is determined by moving a point on a curve $\mathcal{C}$ to $(0,0,0) \in \mathcal{H}_R^3$ and the rotating the curve until the tangent line is in the xz-plane. The resulting normalization equations implicitly defining the corresponding moving frame map are

$$\begin{aligned}
\overline{x} &= x\cos\theta - y\sin\theta + a = 0 \\
\overline{y} &= x\sin\theta + y\cos\theta + b = 0 \\
\overline{z} &= z + c + \frac{1}{2}\left(a\sin\theta - b\cos\theta\right)x + \frac{1}{2}\left(a\cos\theta + b\sin\theta\right)y = 0 \\
\overline{y}' &= \frac{\sin\theta + y'\cos\theta}{\cos\theta - y'\sin\theta} = 0
\end{aligned}$$

The normalization equations $\overline{x} = 0$, $\overline{y} = 0$ and $\overline{y}' = 0$ are identical to the corresponding normalization equations (3.11), (3.12), (3.13) for curves in the Euclidean plane covered in Chapter 3.

Solving the indicated equations for a, b, c and θ follows immediately from the solutions appearing in Chapter 3. The solutions are

$$
\begin{aligned}
a &= -\frac{x+yy'}{\sqrt{1+(y')^2}} \\
b &= \frac{xy'-y}{\sqrt{1+(y')^2}} \\
c &= -z \\
\theta &= \arctan\left(-y'\right).
\end{aligned}
$$

As in the case of curves in the Euclidean plane, expressions for $\cos\theta$ and $\sin\theta$ appearing in elements of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ (5.1) simplify when $\theta = \arctan(-y')$. Using properties of inverse trigonometric functions gives

$$
\cos\theta = \frac{1}{\sqrt{1+(y')^2}} \qquad \textit{and} \qquad \sin\theta = -\frac{y'}{\sqrt{1+(y')^2}}.
$$

Substituting the indicated expressions for a, b, c and θ back into $A \in \mathcal{G}\left(\mathcal{H}_R^3\right)$ gives the moving frame map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R}, \mathbb{R}^2\right) \to \mathcal{G}\left(\mathcal{H}_R^3\right)$:

$$
\rho(x, y, z, y', z') = \begin{pmatrix} \frac{1}{\sqrt{1+(y')^2}} & \frac{y'}{\sqrt{1+(y')^2}} & 0 & -\frac{x+yy'}{\sqrt{1+(y')^2}} \\ -\frac{y'}{\sqrt{1+(y')^2}} & \frac{1}{\sqrt{1+(y')^2}} & 0 & \frac{xy'-y}{\sqrt{1+(y')^2}} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_R^3\right).
$$

For a particular curve $\mathcal{C}$ given by $(x, y(x), z(x))$, the moving frame is the restriction of ρ to the jet of $\mathcal{C}$: $\rho : \mathrm{j}^{(n)}(\mathcal{C}) \to \mathcal{G}\left(\mathcal{H}_R^3\right)$.

5.2.3 Invariantization

Substituting the moving frame components into (5.7), we find the first order differential invariant

$$
i\left(z'\right) = \frac{2z' + y - xy'}{2\sqrt{1+(y')^2}}. \tag{5.9}
$$

Expressions for differential invariants of second order can be found by making the appropriate substitutions into (5.8).

- $i\left(y''\right) = \dfrac{y''}{\left(1+(y')^2\right)^{3/2}}$

- $i\left(z''\right) = \dfrac{-y''\left(x + yy' + 2y'z'\right) + 2z''\left(1+(y')^2\right)}{2\left(1+(y')^2\right)^2}$

Remark 5.2.1. *Note that the invariants obtained from the invariantization of the derivatives y'', $y^{(3)}$, $y^{(4)}$, etc. will be invariants of curves in the Euclidean plane. This is because the action of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ on $\mathcal{H}_R^3 \sim \mathbb{R}^3$ projects onto the action of $\mathcal{G}\left(\mathbb{E}^2\right)$ on $\mathbb{E}^2$ and the prolonged differentiation operators $\mathcal{D}_x$ are identical.*

5.3 Surfaces

We look at surfaces $\mathcal{S}$ in $\mathbb{R}^3$ where $z = z(x,y)$ is a function of the independent variables x and y. On $\mathcal{J}^{(n)}\left(\mathcal{S},\mathbb{R}^3\right)$, we are then considering jets of surfaces with points of the form $\mathbf{z}^{(n)} = (x, y, z, z_x, z_y, \ldots z_K,) \in \mathcal{J}^{(n)}\left(\mathbb{R}^2,\mathbb{R}^1\right)$, where K runs over partial derivative strings of length less than or equal to n. For example, $\mathbf{z}^{(1)} = (x, y, z, z_x, z_y)$ on $\mathcal{J}^{(1)}\left(\mathbb{R}^2,\mathbb{R}^1\right)$.

5.3.1 Prolonged Transformations

We begin by finding the prolonged transformation operators described in Definition 16. The required derivatives of (5.2) and (5.3) are

$$\mathbb{J} = \begin{pmatrix} D_x\overline{x} & D_y\overline{x} \\ D_x\overline{y} & D_y\overline{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The corresponding prolonged differential operators are

$$\begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{pmatrix} = \mathbb{J}^{-T} \begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix},$$

or

$$\mathcal{D}_x = \cos\theta D_x - \sin\theta D_y \qquad \text{and} \qquad \mathcal{D}_y = \sin\theta D_x + \cos\theta D_y. \tag{5.10}$$

We find the prolonged action of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ to $\mathcal{J}^{(n)}\left(\mathbb{R}^2,\mathbb{R}\right)$ by repeatedly applying $\mathcal{D}_x$ and $\mathcal{D}_y$ to $\overline{z}$. To second order we find

$$\overline{z}_x = \cos\theta z_x - \sin\theta z_y - \frac{b}{2} \tag{5.11}$$
$$\overline{z}_y = \sin\theta z_x + \cos\theta z_y + \frac{a}{2} \tag{5.12}$$
$$\overline{z}_{xx} = \cos^2\theta z_{xx} - 2\cos\theta\sin\theta z_{xy} + \sin^2\theta z_{yy} \tag{5.13}$$
$$\overline{z}_{xy} = \cos\theta\sin\theta z_{xx} + \left(2\cos^2\theta - 1\right) z_{xy} - \sin\theta\cos\theta z_{yy} \tag{5.14}$$
$$\overline{z}_{yy} = \sin^2\theta z_{xx} + 2\cos\theta\sin\theta z_{xy} + \cos^2\theta z_{yy} \tag{5.15}$$

5.3.2 Normalization and The Moving Frame

We find a moving frame for the prolonged action of $\mathcal{G}\left(\mathcal{H}_R^3\right)$ on surfaces by using the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}\left(\mathbb{R}^2,\mathbb{R}^1\right)$ defined by the equations

$$x = 0, \qquad y = 0, \qquad z = 0, \qquad \text{and} \qquad z_y = 0.$$

The cross-section corresponds geometrically to brining a point on a surface $\mathcal{S}$ to $(0,0,0)$ of $\mathcal{H}_R^3$ and then rotating $\mathcal{S}$ so that its tangent plane at $(0,0,0)$ contains the y-axis.

We now find the components of the moving frame map by solving the corresponding normalization equations

$$\begin{aligned}
\overline{x} &= x\cos\theta - y\sin\theta + a = 0\\
\overline{y} &= x\sin\theta + y\cos\theta + b = 0\\
\overline{z} &= z + c + \frac{1}{2}\left(a\sin\theta - b\cos\theta\right)x + \frac{1}{2}\left(a\cos\theta + b\sin\theta\right)y = 0\\
\overline{z}_y &= \sin\theta z_x + \cos\theta z_y + \frac{a}{2} = 0
\end{aligned}$$

for the group parameters a, b, c and θ.

Solving the first three equations we find

$$\begin{aligned}
a &= y\sin\theta - x\cos\theta\\
b &= -\left(x\sin\theta + y\cos\theta\right)\\
z &= -c.
\end{aligned}$$

Substituting $a = y\sin\theta - x\cos\theta$ into the equation for $\overline{z}_y = 0$ gives

$$\overline{z}_y = \sin\theta z_x + \cos\theta z_y + \frac{y\sin\theta - x\cos\theta}{2} = 0.$$

After doing some algebra we have

$$\left(z_x + \frac{y}{2}\right)\sin\theta + \left(z_y - \frac{x}{2}\right)\cos\theta = 0.$$

This gives

$$\tan\theta = \frac{x - 2z_y}{2z_x + y}.$$

We take $\theta = \tan^{-1}\left(\frac{x-2z_y}{2z_x+y}\right)$. With properties of inverse trigonometric functions we have

$$\cos\theta = \frac{2z_x + y}{\sigma} \qquad \textit{and} \qquad \sin\theta = \frac{x - 2z_y}{\sigma},$$

where $\sigma = \sqrt{\left(2z_x + y\right)^2 + \left(2z_y - x\right)^2}$.

Substituting the indicated values of $a, b, c, \cos\theta$ and $\sin\theta$ into an element $A \in \mathcal{G}\left(\mathcal{H}_R^3\right)$, we find the moving frame map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R}^2, \mathbb{R}\right) \to \mathcal{G}\left(\mathcal{H}_R^3\right)$ is given by

$$\rho\left(x, y, z, z_x, z_y\right) = \begin{pmatrix} \frac{2z_x+y}{\sigma} & \frac{2z_y-x}{\sigma} & 0 & \frac{-x(2z_x+y)+y(x-2z_y)}{\sigma}\\ \frac{x-2z_y}{\sigma} & \frac{2z_x+y}{\sigma} & 0 & -\frac{x(x-2z_y)+y(2z_x+y)}{\sigma}\\ \frac{y}{2} & -\frac{x}{2} & 1 & -z\\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_R^3\right). \tag{5.16}$$

Restricted to a surface $\mathcal{S}$ given by the graph of a function $z = z(x, y)$ and its jet, the map $\rho : \mathrm{j}^{(n)}\left(\mathcal{S}\right) \to \mathcal{G}\left(\mathcal{H}_R^3\right)$ becomes a moving frame for the surface $\mathcal{S}$.

5.3.3 Invariantization

With the moving frame map at hand, we can invariantize the higher order derivatives to find differential invariants of all orders. We invariantize the remaining derivatives up to order to two demonstrate. This amounts to substitution of the parameters in the moving frame map (5.16) into (5.11), (5.13), (5.14), and (5.15). To simplify the presentation of the invariants we will let $F = 2z_x + y$ and $G = x - 2z_y$. With these identifications the moving frame is

$$\rho(x, y, z, z_x, z_y) = \begin{pmatrix} \frac{F}{\sigma} & \frac{-G}{\sigma} & 0 & \frac{-xF+yG}{\sigma} \\ \frac{G}{\sigma} & \frac{F}{\sigma} & 0 & -\frac{xG+yF}{\sigma} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_R^3\right).$$

The resulting differential invariants are

- $i(\overline{z}_x) = \dfrac{2Fz_x + Gx + Fy - 2Gz_y}{2\sigma} = \dfrac{1}{2}\sqrt{F^2 + G^2} = \dfrac{1}{2}\sigma$
- $i(\overline{z}_{xx}) = \dfrac{F^2 z_{xx} - 2FGz_{xy} + G^2 z_{yy}}{\sigma^2}$
- $i(\overline{z}_{xy}) = \dfrac{FGz_{xx} + \left(F^2 - G^2\right) z_{xy} - FGz_{yy}}{\sigma^2}$
- $i(\overline{z}_{yy}) = \dfrac{F^2 z_{yy} + 2FGz_{xy} + G^2 z_{xx}}{\sigma^2}$

Chapter 6

Curves and Surfaces in $\mathcal{H}_L^3$

In this chapter we find differential invariants of curves and surfaces in the three-dimensional Heisenberg group $\mathcal{H}_L^3$. The point set is the same point set as the $\mathcal{H}_R^3$, but the geometry is changed as a result of the geometric transformations. This geometry compares to the geometry of the Lorentz-Minkowski plane in the same way that the geometry of $\mathcal{H}_R^3$ compares to the geometry of the Euclidean plane. Comparing the matrices defining the indicated geometries, we see that the 2×2 matrix in the upper left of (6.1) is identical to the 2×2 matrix in the upper left of (4.1), while the 2×2 matrix in the upper left of (5.1) is identical to the 2×2 matrix in the upper left of (3.1). The calculations for finding differential invariants of curves and surfaces in this geometry will be similar in nature to combining the calculations of Chapter 4 and Chapter 5.

6.1 Geometric Transformations

As before we identify a point $(x, y, z) \in \mathbb{R}^3$ with $\mathbf{z} = (x, y, z, 1)^t \in \mathbb{R}^4$. The group of geometric transformations defining the geometry of $\mathcal{H}_L^3$ is

$$\mathcal{G}\left(\mathcal{H}_L^3\right) = \left\{ \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & a \\ \sinh\theta & \cosh\theta & 0 & b \\ \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right) & \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right) & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| a, b, c, \theta \in \mathbb{R} \right\}. \tag{6.1}$$

The action of $A \in \mathcal{G}\left(\mathcal{H}_L^3\right)$ on $\mathbb{R}^3 \sim \mathcal{H}_L^3$ is given by $A\mathbf{z} = \overline{\mathbf{z}} = (\overline{x}, \overline{y}, \overline{z}, 1)^t$, where

$$\overline{x} = x\cosh\theta + y\sinh\theta + a \tag{6.2}$$

$$\overline{y} = x\sinh\theta + y\cosh\theta + b \tag{6.3}$$

$$\overline{z} = z + c + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right)x + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y. \tag{6.4}$$

6.2 Curves

We look at curves $\mathcal{C}$ in $\mathcal{H}_L^3$ where $y = y(x)$ and $z = z(x)$ are functions of the independent variable x. The space of n-jets are analyzed in the Cartesian space

$$\mathcal{J}^{(n)}\left(\mathbb{R}, \mathbb{R}^2\right) = \left\{\left(x, y, z, y', z', y'', z'', \ldots, y^{(n)}, z^{(n)}\right)\right\} \sim \mathbb{R}^{3+2n}.$$

6.2.1 Prolonged Transformations

We we will represent a point in $\mathcal{J}^{(n)}\left(\mathbb{R},\mathbb{R}^2\right)$ by $\mathbf{z}^{(n)} = \left\{\left(x, y, z, y', z', y'', z'', \ldots, y^{(n)}, z^{(n)}\right)\right\}$, where derivatives of the dependent variables y and z are taken with respect to the independent variable x. The implicit differential operator D_x is given by Definition 12. Applying D_x to (6.2) gives

$$D_x\overline{x} = D_x\left(x\cosh\theta + y\sinh\theta + a\right) = \cosh\theta + y'\sinh\theta.$$

The prolonged transformation operator from Definition 13 is then

$$\mathcal{D}_x = \frac{1}{\cosh\theta + y'\sinh\theta}D_x. \tag{6.5}$$

We now apply (6.5) to the transformed coordinates (6.3) and (6.4) to obtain the prolonged action $\mathcal{G}\left(\mathcal{H}_L^3\right)$ of to $\mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$. We have

$$\overline{y}' = \mathcal{D}_x\overline{y} = \frac{\sinh\theta + y_x\cosh\theta}{\cosh\theta + y_x\sinh\theta} \tag{6.6}$$

$$\overline{z}' = \mathcal{D}_x\overline{z} = \frac{z_x + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right) + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y_x}{\cosh\theta + y_x\sinh\theta}. \tag{6.7}$$

We will be able to find a moving frame map for curves in $\mathcal{H}_L^3$ on $\mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$.

The prolongation of the action of $\mathcal{G}\left(\mathcal{H}_L^3\right)$ to higher order jet spaces is obtained by repeated application of $\mathcal{D}_x$:

$$\overline{y}^{(k)} = \mathcal{D}_x^k\left(\overline{y}\right) = \mathcal{D}_x\left(\overline{y}^{(k-1)}\right) \qquad \text{and} \qquad \overline{z}^{(k)} = \mathcal{D}_x^k\left(\overline{z}\right) = \mathcal{D}_x\left(\overline{z}^{(k-1)}\right).$$

Since we will be able to find a moving frame map at order one, we will record the transformed derivatives to order two for the purposes of illustrating the differential invariants. To order two we have

$$\overline{y}'' = \frac{y''}{\left(\cosh\theta + y'\sinh\theta\right)^3} \qquad \textit{and} \qquad \overline{z}'' = \frac{y''\left(a - 2z'\sinh\theta\right) + 2\left(\cosh\theta + y'\sinh\theta\right)z''}{2\left(\cosh\theta + y'\sinh\theta\right)^3}. \tag{6.8}$$

6.2.2 Normalization and The Moving Frame

Comparing (6.2), (6.3), and (6.6) with the transformation laws for $\overline{x}$ (4.4), $\overline{y}$ (4.5), and $\overline{y}'$ (4.7) for curves in the Lorentz-Minkowski plane, we see that they are identical. For the same reasons discussed in Chapter 4, our construction of the moving map will split into cases based on whether $|y'| < 1$ or $|y'| > 1$. The resulting moving frame maps are similar to the corresponding moving frames in Chapter 4.

Case I: $|y'| < 1$

We will start with the case where $|y'| < 1$. We assume this condition holds for all x, otherwise our work applies on open intervals.

We obtain a moving frame for the action of $\mathcal{G}\left(\mathcal{H}_L^3\right)$ on curves by using the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}\left(\mathbb{R},\mathbb{R}^2\right)$ determined by setting $x = 0$, $y = 0$, $z = 0$ and $y' = 0$. This geometrically equivalent to moving a point on the curve $\mathcal{C}$ to $(0,0,0) \in \mathcal{H}_L^3$ and applying a hyperbolic rotation until the tangent line is in the xz-plane.

Using the group transformation laws found in (6.2), (6.3), (6.4), and (6.8), we obtain the normalization equations

$$\begin{aligned}
\overline{x} &= x\cosh\theta + y\sin\theta + a = 0\\
\overline{y} &= x\sinh\theta + y\cosh\theta + b = 0\\
\overline{z} &= z + c + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right)x + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y = 0\\
\overline{y}' &= \frac{\sinh\theta + y'\cos\theta}{\cosh\theta + y'\sinh\theta} = 0.
\end{aligned}$$

Note that the normalization equations $\overline{x} = 0$, $\overline{y} = 0$, $\overline{y}' = 0$ are identical to the normalization equations (4.10), (4.11), (4.12) for curves with $|y'| < 1$ in the Lorentz-Minkowski plane. The solutions to the indicated equations for the group parameters a, b, and θ are

$$a = \frac{yy' - x}{\sqrt{1-(y')^2}}, \qquad b = \frac{xy' - x}{\sqrt{1-(y')^2}}, \qquad \textit{and} \qquad \theta = \tanh^{-1}\left(-y'\right).$$

This also gives

$$\cosh\theta = \frac{1}{\sqrt{1-(y')^2}} \qquad \textit{and} \qquad \sinh\theta = -\frac{y'}{\sqrt{1-(y')^2}}.$$

Substituting the indicated expressions for $a, b, \cosh\theta$ and $\sinh\theta$ into the normalization equation $\overline{z} = 0$ gives

$$c = -z.$$

Substituting the indicated expressions for a, b, c and θ back into $A \in \mathcal{G}\left(\mathcal{H}_L^3\right)$ as in (6.1) gives the moving frame map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R}, \mathbb{R}^2\right) \to \mathcal{G}\left(\mathcal{H}_L^3\right)$:

$$\rho(x,y,z,y',z') = \begin{pmatrix} \frac{1}{\sqrt{1-(y')^2}} & \frac{-y'}{\sqrt{1-(y')^2}} & 0 & \frac{yy'-x}{\sqrt{1-(y')^2}} \\ \frac{-y'}{\sqrt{1-(y')^2}} & \frac{1}{\sqrt{1-(y')^2}} & 0 & \frac{xy'-y}{\sqrt{1-(y')^2}} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_L^3\right) \tag{6.9}$$

Case II: $|y'| > 1$

We now consider the case where $|y'| > 1$. We assume this condition holds for all x, or we work on open intervals.

We obtain a moving frame for the action of $\mathcal{G}\left(\mathcal{H}_L^3\right)$ on curves by using the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}\left(\mathbb{R}, \mathbb{R}^2\right)$ determined by setting $x = 0$, $y = 0$, $z = 0$ and $y' = 2$. The is geometrically equivalent to moving a point on the curve $\mathcal{C}$ to $(0,0,0) \in \mathcal{H}_L^3$ and applying a hyperbolic rotation until $y' = 2$.

Using the group transformation laws found in (6.2), (6.3), (6.4), and (6.8), the normalization equations are

$$\begin{aligned}
\overline{x} &= x\cosh\theta + y\sin\theta + a = 0\\
\overline{y} &= x\sinh\theta + y\cosh\theta + b = 0\\
\overline{z} &= z + c + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right)x + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y = 0\\
\overline{y}' &= \frac{\sinh\theta + y'\cos\theta}{\cosh\theta + y'\sinh\theta} = 2.
\end{aligned}$$

Comparing the above with the normalization equations (4.17), (4.18), (4.19) for curves in the Lorentz-Minkowski plane with $|y'| > 1$, we see that the equations for $\overline{x}$, $\overline{y}$, and $\overline{y}'$ are identical. The solutions to the indicated equations for the group parameters a, b, and θ are

$$a = \frac{x + 2y - (2x+y)y'}{\sqrt{3}\sqrt{-1+(y')^2}}, \qquad b = \frac{2x + y - (x+2y)y'}{\sqrt{3}\sqrt{-1+(y')^2}}, \qquad \textit{and} \qquad \theta = \tanh^{-1}\left(\frac{2-y'}{1-2y'}\right)$$

As in Chapter 4, we have

$$\cosh\theta = \frac{-1+2y'}{\sqrt{3}\sqrt{-1+(y')^2}} \qquad \textit{and} \qquad \sinh\theta = \frac{-2+y'}{\sqrt{3}\sqrt{-1+(y')^2}}.$$

Substituting the indicated expressions for $a, b, \cosh\theta$ and $\sinh\theta$ into the normalization equation $\overline{z} = 0$ gives

$$c = -z.$$

Substituting the indicated expressions for a, b, c and θ back into $A \in \mathcal{G}\left(\mathcal{H}_L^3\right)$ gives the moving frame map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R}, \mathbb{R}^2\right) \to \mathcal{G}\left(\mathcal{H}_L^3\right)$:

$$\rho(x,y,z,y',z') = \begin{pmatrix} \frac{-1+2y'}{\sqrt{3}\sqrt{-1+(y')^2}} & \frac{-2+y'}{\sqrt{3}\sqrt{-1+(y')^2}} & 0 & \frac{x+2y-(2x+y)y'}{\sqrt{3}\sqrt{-1+(y')^2}} \\ \frac{-2+y'}{\sqrt{3}\sqrt{-1+(y')^2}} & \frac{-1+2y'}{\sqrt{3}\sqrt{-1+(y')^2}} & 0 & \frac{2x+y-(x+2y)y'}{\sqrt{3}\sqrt{-1+(y')^2}} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_L^3\right). \tag{6.10}$$

6.2.3 Invariantization

Case I: $|y'| < 1$

Substituting the moving frame components from (6.9) into (6.7) and (6.8) we find the differential invariants up to order two are:

- $i\left(z'\right) = \dfrac{y - xy' + 2z'}{2\sqrt{1-(y')^2}}$
- $i\left(y''\right) = \dfrac{y''}{\left(1-(y')^2\right)^{\frac{3}{2}}}$
- $i\left(z''\right) = \dfrac{\left(-x + yy' + 2y'z'\right)y'' - 2\left(-1+(y')^2\right)z''}{2\left(1-(y')^2\right)}$

Case II: $|y'| > 1$

Substituting the moving frame components from (6.10) into (6.7) and (6.8) we find the differential invariants up to order two are:

- $i\left(z'\right) = \dfrac{\sqrt{3}\left(-y + xy' + 2z'\right)}{2\sqrt{1-\left(y'\right)^2}}$
- $i\left(y''\right) = -\dfrac{3\sqrt{3}y''}{\left(1-\left(y'\right)^2\right)^{\frac{3}{2}}}$
- $i\left(z''\right) = \dfrac{3\left(-x + y\left(-2+y'\right) - 4z' + 2y'\left(x+z'\right)\right)y'' - 6\left(-1+\left(y''\right)^2\right)z''}{2\left(1-\left(y'\right)^2\right)}$

Remark 6.2.1. *Note that the invariants obtained from the invariantization of the derivatives y'', $y^{(3)}$, $y^{(4)}$, etc. will be invariants of curves in the Lorentz-Minkowski plane. This is because the action of $\mathcal{G}\left(\mathcal{H}_L^3\right)$ on $\mathcal{H}_L^3 \sim \mathbb{R}^3$ projects onto the action of $\mathcal{G}\left(\mathbb{L}^2\right)$ on $\mathbb{L}^2$ and the prolonged differentiation operators $\mathcal{D}_x$ are identical.*

6.3 Surfaces

We now look at surfaces $\mathcal{S}$ in $\mathbb{R}^3 \sim \mathcal{H}_L^3$ where $z = z(x,y)$ is a function of the independent variables x and y. As before we are considering jets of surfaces with points of the form $\mathbf{z}^{(n)} = (x, y, z, z_x, z_y, \ldots z_N) \in \mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}^1\right)$. For example, $\mathbf{z}^{(1)} = (x, y, z, z_x, z_y)$ represents a point in $\mathcal{J}^{(1)}\left(\mathbb{R}^2, \mathbb{R}^1\right)$.

6.3.1 Prolonged Transformations

The prolonged transformation operators are described in Definition 16. The required derivatives of (6.2) and (6.3) are

$$\mathbb{J} = \begin{pmatrix} D_x\overline{x} & D_y\overline{x} \\ D_x\overline{y} & D_y\overline{y} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}$$

The corresponding prolonged differential operators are

$$\begin{pmatrix} \mathcal{D}_x \\ \mathcal{D}_y \end{pmatrix} = \mathbb{J}^{-T} \begin{pmatrix} D_x \\ D_y \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} D_x \\ D_y \end{pmatrix},$$

or

$$\mathcal{D}_x = \cosh\theta D_x - \sinh\theta D_y \qquad \text{and} \qquad \mathcal{D}_y = -\sinh\theta D_x + \cosh\theta D_y. \tag{6.11}$$

We find the prolonged action of $\mathcal{G}\left(\mathcal{H}_L^3\right)$ to $\mathcal{J}^{(n)}\left(\mathbb{R}^2, \mathbb{R}\right)$ by repeatedly applying $\mathcal{D}_x$ and $\mathcal{D}_y$ to $\overline{z}$ in (6.4). To second order we have

$$\overline{z}_x = \cosh\theta z_x - \sinh\theta z_y - \frac{b}{2} \tag{6.12}$$

$$\overline{z}_y = -\sinh\theta z_x + \cosh\theta z_y + \frac{a}{2} \tag{6.13}$$

$$\overline{z}_{xx} = \cosh^2\theta z_{xx} - 2\cosh\theta\sinh\theta z_{xy} + \sinh^2\theta z_{yy} \tag{6.14}$$

$$\overline{z}_{xy} = -\cosh\theta\sinh\theta z_{xx} + \left(2\cosh^2\theta + \sinh^2\theta\right) z_{xy} - \cosh\theta\sinh\theta z_{yy} \tag{6.15}$$

$$\overline{z}_{yy} = \sinh^2\theta z_{xx} - 2\cosh\theta\sinh\theta z_{xy} + \cosh^2\theta z_{yy} \tag{6.16}$$

6.3.2 Normalization and The Moving Frame

As was the case for curves, we will need to construct two different moving frame maps for surfaces. The cases split depending on the quantity $\left|\dfrac{2z_y - x}{2z_x + y}\right|$ and whether we can solve $\overline{z}_y = 0$ or $\overline{z}_x = 0$ for the group parameters.

Case I: $\left|\dfrac{2z_y - x}{2z_x + y}\right| < 1$ **and Solving** $\overline{z}_y = 0$

We first consider the case where we set $z_y = 0$ in the normalization equations. We use the cross-section $\mathcal{K}^{(1)} \subset \mathcal{J}^{(1)}\left(\mathbb{R}^2, \mathbb{R}\right)$ defined by

$$x = 0, \qquad y = 0, \qquad z = 0, \qquad \textit{and} \qquad z_y = 0. \tag{6.17}$$

The cross-section corresponds to moving a point on the surface to $(0, 0, 0)$ and then applying a hyperbolic rotation to bring the normal vector of the surface into the xz-plane. The resulting normalization equations are

$$\begin{aligned}
\overline{x} &= x\cosh\theta + y\sinh\theta + a = 0 \\
\overline{y} &= x\sinh\theta + y\cosh\theta + b = 0 \\
\overline{z} &= z + c + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right)x + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y = 0 \\
\overline{z}_y &= -\sinh\theta z_x + \cosh\theta z_y + \frac{a}{2} = 0.
\end{aligned}$$

The equations $\overline{x} = 0$ and $\overline{z}_y = 0$ can be solved for a and θ. The normalization equation $\overline{x} = 0$ gives

$$a = -\left(x\cosh\theta + y\sinh\theta\right).$$

Substitution into $\overline{z}_y = 0$ and some algebra gives

$$-\left(z_x + \frac{y}{2}\right)\sinh\theta + \left(z_y - \frac{x}{2}\right)\cosh\theta = 0.$$

Algebra and properties of hyperbolic trig functions then give

$$\theta = \tanh^{-1}\left(\frac{2z_y - x}{2z_x + y}\right).$$

With the expression for θ, we then find simplified expressions for $\cosh\theta$ and $\sinh\theta$:

$$\cosh\theta = \frac{2z_x + y}{\sigma} \qquad \textit{and} \qquad \sinh\theta = \frac{2z_y - x}{\sigma},$$

where $\sigma = \sqrt{(2z_x + y)^2 - (2z_y - x)^2}$.

With expressions for $\cosh\theta$ and $\sinh\theta$, we can now find a, b, and c from the normalization equations:

$$\begin{aligned} a &= -\frac{2\left(xz_x + yz_y\right)}{\sigma} \\ b &= \frac{x\left(x - 2z_y\right) - y\left(2z_x + y\right)}{\sigma} \\ c &= -z \end{aligned}$$

Making substitutions for the indicated expressions of $a, b, c, \cosh\theta$, and $\sinh\theta$ into $A \in \mathcal{G}\left(\mathcal{H}_L^3\right)$ as given in (6.1), we find the moving map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R}^2, \mathbb{R}\right) \to \mathcal{G}\left(\mathcal{H}_L^3\right)$ to be

$$\rho\left(x, y, z, z_x, z_y\right) = \begin{pmatrix} \frac{2z_x+y}{\sigma} & \frac{2z_y-x}{\sigma} & 0 & -\frac{2(xz_x+yz_y)}{\sigma} \\ \frac{2z_y-x}{\sigma} & \frac{2z_x+y}{\sigma} & 0 & \frac{x(x-2z_y)-y(2z_x+y)}{\sigma} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_L^3\right). \tag{6.18}$$

As usual, restricted to a surface $\mathcal{S}$ given by the graph of a function $z = z(x, y)$ and its jet, the map $\rho : \mathrm{j}^{(n)}\left(\mathcal{S}\right) \to \mathcal{G}\left(\mathcal{H}_L^3\right)$ becomes a moving frame for the surface $\mathcal{S}$.

Case II: $\left|\dfrac{2z_x + y}{2z_y - x}\right| < 1$ **and Solving** $\overline{z}_{\overline{x}} = 0$

In the second case we set $z_x = 0$ in the cross-section defining the moving frame and the normalization equations. The cross-section is

$$x = 0, \qquad y = 0, \qquad z = 0, \qquad \textit{and} \qquad \overline{z}_x = 0. \tag{6.19}$$

The cross-section corresponds to moving a point on the surface to $(0, 0, 0)$ and then applying a hyperbolic rotation to bring the normal vector into the yz-plane. The resulting normalization equations are

$$\begin{aligned} \overline{x} &= x\cosh\theta + y\sinh\theta + a = 0 \\ \overline{y} &= x\sinh\theta + y\cosh\theta + b = 0 \\ \overline{z} &= z + c + \frac{1}{2}\left(a\sinh\theta - b\cosh\theta\right)x + \frac{1}{2}\left(a\cosh\theta - b\sinh\theta\right)y = 0 \\ \overline{z}_x &= \cosh\theta z_x - \sinh\theta z_y - \frac{b}{2}. \end{aligned}$$

The required algebra to solve the normalization equations is near identical to the first case, except we begin by solving the equations $\overline{y} = 0$ and $\overline{z}_x = 0$ for b and θ.

The normalization equation $\overline{y} = 0$ gives

$$b = -\left(x \sinh\theta + y\cosh\theta\right).$$

Substitution into $\overline{z}_x = 0$ and some algebra gives

$$-\left(z_y - \frac{x}{2}\right)\sinh\theta + \left(z_x + \frac{y}{2}\right)\cosh\theta = 0.$$

Algebra and properties of hyperbolic trig functions then give

$$\theta = \tanh^{-1}\left(\frac{2z_x + y}{2z_y - x}\right).$$

With the expression for θ, we then find simplified expressions for $\cosh\theta$ and $\sinh\theta$:

$$\cosh\theta = \frac{2z_y - x}{\beta} \qquad \textit{and} \qquad \sinh\theta = \frac{2z_x + y}{\beta},$$

where $\beta = \sqrt{\left(2z_y - x\right)^2 - \left(2z_x + y\right)^2}$. With expressions for $\cosh\theta$ and $\sinh\theta$, we can now solve the remaining normalization equations for a, b, and c:

$$\begin{aligned} a &= -\frac{2\left(xz_x + yz_y\right)}{\sigma} \\ b &= \frac{x\left(x - 2z_y\right) - y\left(2z_x + y\right)}{\sigma} \\ c &= -z \end{aligned}$$

Making substitutions for the indicated expressions of $a, b, c, \cosh\theta$, and $\sinh\theta$ into $A \in \mathcal{G}\left(\mathcal{H}_L^3\right)$ as given in (6.1), we find the moving map $\rho : \mathcal{J}^{(1)}\left(\mathbb{R}^2, \mathbb{R}\right) \to \mathcal{G}\left(\mathcal{H}_L^3\right)$ to be

$$\rho\left(x, y, z, z_x, z_y\right) = \begin{pmatrix} \frac{2z_y - x}{\beta} & \frac{2z_x + y}{\beta} & 0 & \frac{x(x - 2z_y) - y(2z_x + y)}{\beta} \\ \frac{2z_x + y}{\beta} & \frac{2z_y - x}{\beta} & 0 & -\frac{2(xz_x + yz_y)}{\beta} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}\left(\mathcal{H}_L^3\right). \tag{6.20}$$

The moving frame map for a surface $\mathcal{S}$ given by the graph of a function $z = z(x, y)$ is then $\rho : \mathrm{j}^{(n)}\left(\mathcal{S}\right) \to \mathcal{G}\left(\mathcal{H}_L^3\right)$.

6.3.3 Invariantization

Finally, we find the differential invariants up to order two for surfaces in $\mathcal{H}_L^3$. In each case, this amounts to substituting the components of the moving frame map into the appropriate expressions from (6.12)–(6.16). Note that in first case, the invariant $i(z_y)$ is trivial (we normalized $\overline{z}_{\overline{y}} = 0$), while in the second case the invariant $i(z_x) = 0$. For purposes of simplifying the expressions of the invariants, we let $F = 2z_x + y$ and $G = 2z_y - x$.

With the indicated simplifications, the respective moving frames are

$$\rho\left(x, y, z, z_x, z_y\right) = \begin{pmatrix} \frac{F}{\sigma} & \frac{G}{\sigma} & 0 & -\frac{xF + yG}{\sigma} \\ \frac{G}{\sigma} & \frac{F}{\sigma} & 0 & -\frac{xG + yF}{\sigma} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$\rho\left(x,y,z,z_x,z_y\right)=\begin{pmatrix} \frac{G}{\beta} & \frac{F}{\beta} & 0 & -\frac{xG+yF}{\beta} \\ \frac{F}{\beta} & \frac{G}{\beta} & 0 & -\frac{xF+yG}{\beta} \\ \frac{y}{2} & -\frac{x}{2} & 1 & -z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Case I: $\left|\dfrac{2z_y-x}{2z_x+y}\right|<1$ **and** $\overline{z}_{\overline{y}}=0$

- $i\left(z_x\right)=\dfrac{xG+yF+2Fz_x-2Gz_y}{2\sigma}=\dfrac{xG+yF+2Fz_x-2Gz_y}{2\sqrt{F^2-G^2}}$
- $i\left(z_{xx}\right)=\dfrac{G^2z_{yy}-2FGz_{yy}+F^2z_{xx}}{\sigma^2}=\dfrac{G^2z_{yy}-2FGz_{yy}+F^2z_{xx}}{F^2-G^2}$
- $i\left(z_{xy}\right)=\dfrac{-FGz_{yy}+(F^2+G^2)z_{xy}-FGz_{xx}}{\sigma^2}=\dfrac{-FGz_{yy}+(F^2+G^2)z_{xy}-FGz_{xx}}{F^2-G^2}$
- $i\left(z_{yy}\right)=\dfrac{F^2z_{yy}-2FGz_{xy}+G^2z_{xx}}{\sigma^2}=\dfrac{F^2z_{yy}-2FGz_{xy}+G^2z_{xx}}{F^2-G^2}$

Case II: $\left|\dfrac{2z_x+y}{2z_y-x}\right|<1$ **and Solving** $\overline{z}_{\overline{x}}=0$

- $i\left(z_y\right)=\dfrac{-xG-yF-2Fz_x+2Gz_y}{2\beta}=\dfrac{-xG-yF-2Fz_x+2Gz_y}{2\sqrt{G^2-F^2}}$
- $i\left(z_{xx}\right)=\dfrac{F^2z_{yy}-2FGz_{yy}+G^2z_{xx}}{\beta^2}=\dfrac{F^2z_{yy}-2FGz_{yy}+G^2z_{xx}}{G^2-F^2}$
- $i\left(z_{xy}\right)=\dfrac{-FGz_{yy}+(F^2+G^2)z_{xy}-FGz_{xx}}{\beta^2}=\dfrac{-FGz_{yy}+(F^2+G^2)z_{xy}-FGz_{xx}}{G^2-F^2}$
- $i\left(z_{yy}\right)=\dfrac{G^2z_{yy}-2FGz_{yy}+F^2z_{xx}}{\beta^2}=\dfrac{G^2z_{yy}-2FGz_{yy}+F^2z_{xx}}{G^2-F^2}$

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